UNITED STATES
NAVAL POSTGRADUATE SCHOOL

ELECTRON BALLISTICS
AND
ELECTROMAGNETIC WAVES IN
THE IONOSPHERE

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ABSTRACT:

It is shown that ion collisions introduce a slight reduction in the plasma frequency, along with an exponential decay of transient electron oscillations. The critical frequencies for penetration of a homogeneous ionosphere, for both isotropic and anisotropic ionospheres, are determined. The characteristic waves, for electromagnetic propagation within a homogeneous anisotropic ionosphere, are developed by considering an infinite series of electron velocities, produced by Ampere's force law, reacting with the electron oscillation produced by an exciting electric field. The complex indices of refraction are determined, both from a dispersion equation and from a derivation of the Appleton equation, which uses an arbitrary selection of the coordinate axes, thus emphasizing the invariance of the Appleton equation. Vector and tensor algebra is used throughout the analytical developments.

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Symbols

$\bar{E}_o$ = Vector magnitude of incident field

$\bar{B}_o$ = Geodesic field vector

$\bar{k}_o = \omega/c$ = Free space wave number

μ' = Relative permeability constant

ϵ_c' = Complex relative dielectric constant

$\bar{\bar{\epsilon}}'$ = Complex relative dielectric tensor

$\bar{n}$ = Complex vector index of refraction

$\bar{\gamma} = j\, k_o\, \bar{n}$ = Complex vector propagation constant

ψ = Angle between $\bar{\gamma}$ and $\bar{B}_o$

$\zeta = \text{Cos}^{-1}\ \bar{a}_\theta \cdot \bar{a}_z$

$x,\ y,\ Z$ = Cartesian coordinates

$r,\ \theta, \varphi$ = Spherical coordinates

ξ = Small displacement along X-axis

$\bar{F}$ = Force vector

e = Positive magnitude of electron charge

m = Mass of electron

$-\ e/m = -1.77 \times 10^{11}$ coul/Kg

ν = Average number of electron-ion collisions per second

$\bar{r}$ = Radius vector

$\bar{V}$ = Vector velocity of an electron

N = Density of electrons per cubic meter

$\mu_o = 4\ \pi \times 10^{-7}$ h/m = Permeability of free space

$\epsilon_o \approx \frac{1}{36\pi} \times 10^{-9}$ fd/m = Permitivity of free space

$\omega_p = \sqrt{\frac{Ne^2}{m\, \epsilon_o}}$ = Resonant angular frequency, lossless plasma

$\omega_b = \frac{eB_o}{m}$ = Cyclotron, or gyro, frequency

$\bar{E},\ \bar{H}$ = Electromagnetic field components

Symbols (Continued)

$\bar{D}$ = Electric flux density vector

$\bar{\iota}$ = Current density in amps. per sq. m.

ρ_s = Surface charge density in coul. per sq. m.

ρ = Charge density in coul. per cubic m.

$X = \omega_p{}^2/\omega^2$

$Y = \omega_b/\omega = \sqrt{Y_x^2 + Y_y^2 + Y_z^2}$

$Y_x = Y\cos\theta$, $Y_y = Y\sin\theta\cos\varphi$, $Y_z = Y_L = Y\sin\theta\sin\varphi = Y\cos\psi$

$Y_T = \sqrt{Y_x^2 + Y_y^2} = \sqrt{Y^2 - Y_L^2} = Y\sin\psi$

$\Gamma = 1 - j\,\nu/\omega$

$M^2 = n^2 - 1 = \bar{n}\,.\,\bar{n} - 1$

$\bar{a}_i$ = Unit vectors

$\bar{A}_1 = \bar{a}_\theta \times \bar{a}_z = \bar{a}_1 \sin\zeta$

$\bar{A}_2 = (\bar{a}_\theta \times \bar{a}_z)\ \bar{a}_z = \bar{a}_2 \sin\zeta$

$\tilde{X} = 1 - \dfrac{\Gamma X}{\Gamma^2 - Y^2}$

$\tilde{Y} = \dfrac{X\ Y}{\Gamma^2 - X^2}$

$\tilde{Z} = 1 - \dfrac{X}{\Gamma}$

$E_{xy} = \sqrt{E_x{}^2 + E_y{}^2}$, $\phi_{xy} = \tan^{-1}\dfrac{E_y}{E_x}$

ω_c = Critical frequency

ω_{c_o} = Ordinary wave critical frequency

ω_{cL} = Left circularly polarized wave critical frequency

ω_{cR} = Right circularly polarized wave critical frequency

$\bar{\bar{\sigma}}$ = Lorentz conductivity tensor

$\bar{\bar{\rho}}$ = Lorentz resistivity tensor

Symbols (Continued)

$\bar{\bar{I}}$ = Identity tensor

s = Length of path through the ionosphere

$c \approx 3 \times 10^8$ m/sec = Speed of light in a vacuum

V_g = Group speed

V_p = Phase speed

α = Attenuation constant

β = Phase constant

θ_o = Angle of incidence

θ_r = Angle of refraction

R = Measure of polarization

$n_{ij} \equiv n_i n_j$ = Components of refractive tensor

1. PLASMA FREQUENCY

1.1 Lossless Case

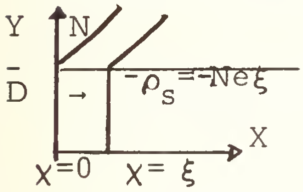


Figure 1

Consider a lossless plasma having a density N $(\frac{elec}{m^3})$ of free electrons. If the electrons are compressed laterally through a small displacement, ξ, along the X-axis, a surface charge density

$$-\rho_s = -Ne\,\xi \tag{1}$$

may be considered to exist at $\chi = \xi$. Then, from the definition of the electric displacement vector, from $\chi = o$ to $\chi = \xi$,

$$D_\chi = \rho_s. \tag{2}$$

Hence,

$$D_\chi = \epsilon_o E_\chi = Ne\,\xi\,, \tag{3}$$

or the force F_χ is given by,

$$F_\chi = -eE_\chi = -\frac{Ne^2}{\epsilon_o}\,\xi. \tag{4}$$

Since force is given by the product of mass with accelleration,

$$m\frac{d^2\xi}{dt^2} + \frac{Ne^2}{\epsilon_o}\,\xi = 0. \tag{5}$$

Substituting $\omega_p^{\;2} = \frac{Ne^2}{m\,\epsilon_o}$ into (5), the harmonic equation for free oscillations of the electrons is obtained, namely,

$$\frac{d^2\xi}{dt^2} + \omega_p^{\;2}\,\xi = 0. \tag{6}$$

The solution of (6) is sinusoidal with an angular frequency ω_p. Thus ω_p is called the resonant angular frequency for the lossless plasma.

1.2 Lossy Case

Suppose there are an average number, ν, of collisions per second between electrons and positive ions. This introduces an average loss of momentum, or an equivalent fluid resistance term, into equation (5), which equation becomes,

$$m\frac{d^2\xi}{dt^2} + \nu m\frac{d\xi}{dt} + \frac{Ne^2}{\epsilon_o}\,\xi = 0. \tag{7}$$

Dividing by m, and using the operational method of substituting $\xi = \xi_o e^{pt}$

into equation (7), the determinantal equation is obtained,

$$(p^2 + \nu p + \omega_p^{\ 2})\ \xi = 0. \tag{8}$$

Equation (8) yields the characteristic roots,

$$p = -\nu/2 \pm j\ \omega_p \sqrt{1 - (\nu/(2\ \omega_p))^2} \tag{9}$$

From equation (9), it may be concluded that ionic collisions in the plasma introduce a time decay constant of $2/\nu$, and reduce the plasma frequency by a factor $\sqrt{1 - (\nu/(2\ \omega_p))^2}$. For a low loss plasma, the latter factor usually is ignored.

2. HOMOGENEOUS ISOTROPIC IONOSPHERE

2.1 Low Loss Case

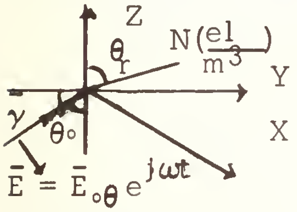


Figure 2

Consider a small collision frequency for a homogeneous isotropic ionosphere of electron density $N(\frac{el}{m^3})$. Let a uniform plane wave front, $\bar{E}_{\circ\theta} e^{j\omega t}$, be incident upon the ionosphere at an angle of incidence $\theta_\circ$. If $\bar{r}$ is a radius vector describing the path of motion of an electron having a velocity $\bar{V}$, the force $\bar{F}$ upon an electron is given by,

$$\bar{F} = -e\bar{E} = m\ \frac{d^2\ \bar{r}}{dt^2} + \nu\ m\ \frac{d\ \bar{r}}{dt}\ , \tag{10}$$

or

$$\frac{d\ \bar{V}}{dt} + \nu\ \bar{V} = -\frac{e}{m}\ \bar{E}\ . \tag{11}$$

Substituting $\bar{V}\ e^{j\omega t}$ into (11) and solving for the steady state velocity,

$$\bar{V} = \frac{-\ e\ \bar{E}}{m\ (j\omega + \nu)} = \frac{-\ e\ \bar{E}}{jm(\ \omega - j\ \nu)}\ . \tag{12}$$

The equivalent current density due to the motion of electrons then becomes,

$$\bar{\iota} = \rho\ \bar{V} = -Ne\bar{V} = \frac{Ne^2\ \bar{E}}{m\ (\nu + j\omega)}\ . \tag{13}$$

Equations (12) and (13) show that the electron path, velocity, and current density are linear in the direction of $\bar{E}$.

In order to see the effects upon the electromagnetic wave, Maxwell's equations in the steady state phasor form will be used.

The Maxwell's equations are, div $\bar{B} = 0$, div $\bar{D} = 0$, curl $\bar{E} = -\dfrac{\partial}{\partial t}\bar{B}$,

$$\text{curl}\,\bar{H} = \bar{\iota} + \frac{\partial \bar{D}}{\partial t}\ ,\ \bar{B} = \mu' \mu_\circ \bar{H}\ ,\ \bar{D} = \epsilon' \epsilon_\circ \bar{E}\ . \tag{14}$$

The corresponding complex form is,

$$\begin{aligned} &\nabla \cdot \bar{H} = 0,\ \nabla \cdot \bar{E} = 0,\ \nabla \times \bar{E} = -j\omega\mu_\circ \bar{H}, \\ &\nabla \times \bar{H} = \bar{\iota} + j\omega\ \epsilon_\circ \bar{E}. \end{aligned} \tag{15}$$

Substituting from equation (13),

$$\begin{aligned} &\nabla \times \bar{H} = \frac{Ne^2\ \bar{E}}{m\ (\nu + j\omega)} + j\omega\ \epsilon_\circ \bar{E} = j\omega\epsilon_\circ\ x \\ &[1 + \frac{Ne^2}{j\omega\,\epsilon_\circ\, m(\nu + j\omega)}]\ \bar{E}\ , \end{aligned} \tag{16}$$

or,

$$\begin{aligned} &\nabla \times \bar{H} = j\omega\ \epsilon_\circ \left[1 + \frac{\omega_p{}^2\ (\nu - j\omega)}{j\omega(\nu^2 + \omega^2\ )} \right] \bar{E} = \\ &j\omega\ \epsilon_\circ \left[(\ 1 - \frac{\omega_p{}^2}{\nu^2 + \omega^2}) - j\,\frac{\nu\ \omega_p{}^2}{\omega(\nu^2 + \omega^2)} \right] \bar{E}\ , \end{aligned} \tag{17}$$

in which $\omega_p{}^2 = \dfrac{Ne^2}{m\ \epsilon_\circ}$ was also substituted in the right member of equation (17).

From equation (17), an equivalent complex dielectric constant ϵ_c' is obtained,

$$\epsilon_c' \equiv \epsilon\ ' - j\ \epsilon'' = (\ 1 - \frac{\omega_p{}^2}{\omega^2 + \nu^2}) - j\,\frac{\nu\ \omega^2}{\omega^2 + \nu^2}\ . \tag{18}$$

Returning to euqations (15), the wave equation will be determined by elimination.

$$\begin{aligned} &\nabla \times (\ \nabla \times \bar{E}) \equiv \nabla(\ \nabla \cdot \bar{E}) - \nabla^2\,\bar{E} = -\nabla^2\,\bar{E} = -j\omega\mu_\circ x \\ &\nabla \times \bar{H}\ . \end{aligned} \tag{19}$$

Using $\gamma \equiv \alpha + j\beta = jk_\circ \sqrt{\epsilon_c'}$, and $k_\circ = \omega\sqrt{\mu_\circ \epsilon_\circ} = \omega/c$, upon substituting for $\nabla \times \bar{H}$, equation (19) becomes,

$$\nabla^2\,\bar{E} + k_\circ^2\ \epsilon_c'\ \bar{E} = 0. \tag{20}$$

For $\nu \approx o$, $n \equiv \sqrt{\epsilon_c'}$ will be evaluated.

$$\begin{aligned} &n = (\epsilon' - j\epsilon'')^{1/2} = \sqrt{\epsilon\ '}\ (\ 1 - j\,\frac{\epsilon''}{\epsilon'})^{1/2} \\ &\approx \sqrt{\epsilon'}\ (\ 1 - j\,\frac{\epsilon''}{2\ \epsilon'}) = \sqrt{\epsilon\ '} - j\,\frac{\epsilon'}{2\sqrt{\epsilon'}} \end{aligned} \tag{21}$$

and hence,

$$\gamma = j\, k_o n \approx \frac{k_o\, \epsilon''}{2\sqrt{\epsilon'}} + j\, k_o \sqrt{\epsilon'} \quad . \qquad (22)$$

From equation (18) with $\nu \approx 0$,

$$\gamma \approx \frac{\nu\, k_o}{2\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}} + j\, k_o \sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2} \qquad (23)$$

The phase velocity is,

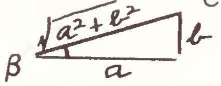
$$V_p \equiv \frac{\omega}{\beta} = \frac{\omega / k_o}{\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}} = \frac{c}{\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}} \qquad (24)$$

and the group velocity is,

$$V_g \equiv \frac{d\,\omega}{d\,\beta} = c\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2} \qquad (25)$$

2.2 Lossy Case

For the lossy case, the exact value of $n = \sqrt{\epsilon'_c}$ should be used. For this purpose, consider $\sqrt{a \pm j\, b}$,



, $B = \tan^{-1} b/a$.

$$\sqrt{a \pm j\, b} = \sqrt[4]{a^2+b^2}\; e^{\pm j\, B/2}$$

$$\cos B/2 = \sqrt{\frac{1+\cos B}{2}} = \frac{1}{\sqrt{2}}\sqrt{1+\frac{a}{\sqrt{a^2+b^2}}} \; ,$$

$$\sin B/2 = \frac{1}{\sqrt{2}}\sqrt{1-\frac{a}{\sqrt{a^2+b^2}}} \; .$$

$$\sqrt{a \pm j\, b} = \frac{1}{\sqrt{2}}\sqrt{\frac{2}{\sqrt{a^2+b^2+a}}} \pm$$

$$j\,\frac{1}{\sqrt{2}}\sqrt{\sqrt{a^2+b^2}-a} \quad . \qquad (26)$$

Substituting from equation (18) into (26),

$$n = \frac{1}{\sqrt{2}}\left\{\left[\left(1-\frac{\omega_p^2}{\omega^2+\nu^2}\right)^2 + \frac{\omega_p^4\,\nu^2}{\omega^2(\omega^2+\nu^2)^2}\right]^{1/2} + \left(1-\frac{\omega_p^2}{\omega^2+\nu^2}\right)\right\}^{1/2} - j\,\frac{1}{\sqrt{2}}\left\{\left[\left(1-\frac{\omega_p^2}{\omega^2+\nu^2}\right)^2 + \frac{\omega_p^4\,\nu^2}{\omega^2(\omega^2+\nu^2)^2}\right]^{1/2} - \left(1-\frac{\omega_p^2}{\omega^2+\nu^2}\right)\right\}^{1/2} \qquad (27)$$

Thus, $\gamma = k_o \mathcal{Im}\,[n]$ and $\beta = k_o \mathrm{Re}\,[n]$. (28)

The Re [n] is also spoken of as the refractive index for the ordinary wave.

2.21 Critical Frequency

From equation (28), the phase velocity within the ionosphere is,

$$V_p = \frac{1}{\sqrt{\mu_o \epsilon' \epsilon_o}} = \frac{c}{\mathrm{Re}\,[n]} \tag{29}$$

Designating the angle of refraction by θ_r, from Snell's law,

$$\sin \theta_r = \frac{\sin \theta_o}{\mathrm{Re}\,[n]} \tag{30}$$

If $\theta_r < \pi/2$, the wave enters the ionosphere, is refracted, and passes through the ionosphere, unless the right member of (30) is greater than unity.

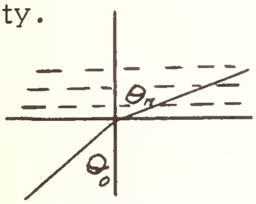


Figure 3

If the right member of (30) is greater than unity, no real angle θ_r exists and the wave is reflected. Designating the critical angular frequency by ω_c, this frequency is determined by $\theta_r = \pi/2$. Thus, for any angle of incidence, the critical frequency is determined from,

$$\sin \theta_o = \mathrm{Re}\,[n]. \tag{31}$$

For the low loss case, from equations (23) and (31),

$$\sin \theta_o = \sqrt{1 - \left(\frac{\omega_p}{\omega_c}\right)^2}, \tag{32}$$

or

$$\omega_c = \omega_p \sec \theta_o \tag{33}$$

Equation (33) is known as the secant law, and yields $\omega_c = \omega_p$ for normal incidence. Substituting the numerical values of e, m, and ϵ_o into (33),

$$f_c = 9\sqrt{N} \sec \theta_o, \tag{34}$$

in which f_c is in cycles per second. If the electron density is given per cubic centimeter, the f_c will be in kilocycles per second.

At a given angle of incidence, frequencies $f > f_c$ will pass through the ionosphere, and frequencies $f < f_c$ will be reflected.

The preceding analysis has ignored the earth's geodesic field. All waves acting accordingly are termed ordinary waves.

3. HOMOGENEOUS ANISOTROPIC IONOSPHERE

3.0 Direction Cosines

In order to take into consideration the effects of the geodesic field upon an electromagnetic wave propagating in the ionosphere, various vector directions must be taken into consideration. Hence, a table of direction cosines is necessary. Such a table may be formulated either by projections or by spherical trigonometry, or by a combination of both methods. Referring to the figure:

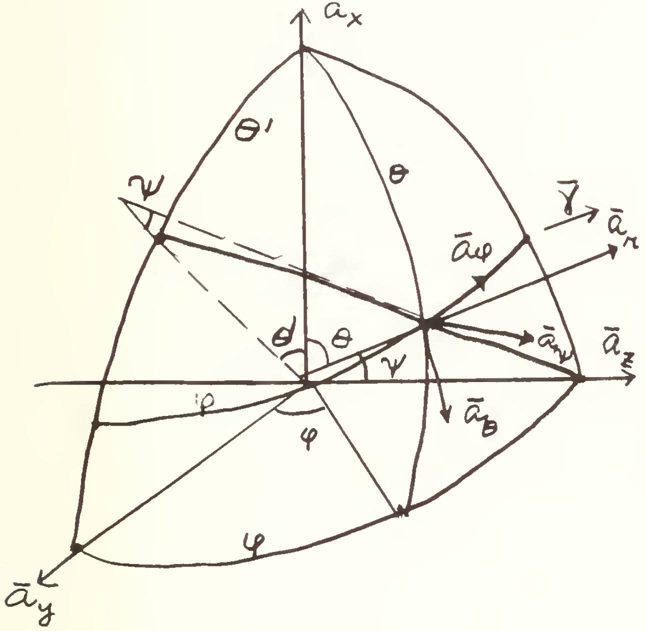


Figure 4

$$\cos\psi = \cos\theta\cos\pi/2 + \sin\theta \times \sin\pi/2 \cos(\pi/2 - \varphi)$$

$$\cos\psi = \sin\theta\sin\varphi$$

$$\bar{a}_r \cdot \bar{a}_z = \cos\psi = \sin\theta\sin\varphi$$

$$-\bar{a}_\psi \cdot \bar{a}_z = \sin\psi$$

$$-\bar{a}_\psi \cdot \bar{a}_y = -\cos\psi\sin\theta'$$

$$-\bar{a}_\psi \cdot \bar{a}_x = -\cos\psi\cos\theta'$$

$$\cos\theta = \cos\psi\cos\pi/2 + \sin\psi \times \sin\pi/2 \cos\theta'$$

$$\cos\theta' = \frac{\cos\theta}{\sin\psi}$$

$$\sin\theta' = \sqrt{1 - \frac{\cos^2\theta}{\sin^2\psi}} =$$

$$\sqrt{\frac{1 - \sin^2\theta\sin^2\varphi - \cos^2\varphi}{\sin^2\psi}} =$$

$$\frac{\sin\theta\cos\varphi}{\sin\psi}$$

$$\sin\psi = \sqrt{1 - \sin^2\theta\sin^2\varphi}$$

From the above, the following table may be compiled:

	$\bar{a}_X$	$\bar{a}_y$	$\bar{a}_z$	$-\bar{a}_\psi$
$\bar{a}_r$	$\cos\theta$	$\sin\theta\cos\varphi$	$\sin\theta\sin\varphi$	0
$\bar{a}_\theta$	$-\sin\theta$	$\cos\theta\cos\varphi$	$\cos\theta\sin\varphi$	$\frac{\cos\theta\sin\varphi}{\sin\psi}$
a_φ	0	$-\sin\varphi$	$\cos\varphi$	$\frac{\cos\varphi}{\sin\psi}$
$-\bar{a}_\psi$	$\frac{-\sin\theta\cos\theta\sin\varphi}{\sin\psi}$	$\frac{-\sin^2\theta\sin\varphi\cos\varphi}{\sin\psi}$	$\sin\psi$	1

(35)

3.1 Electron Current Density

Assume the Z-axis of a coordinate system to lie along the geodesic field in the ionosphere, and assume an electromagnetic field, $\bar{E} = E_0\,\bar{a}_\theta$, to be incident at angle θ_0 upon the ionosphere. Equation (13) gives the current density, due to the presence of the electromagnetic field along with the free electrons. However, it ignores the presence of the geodesic field, the geodesic field being $B_0\,\bar{a}_z$.

The zero order component, $\bar{\iota}_0$ of equation (13), reacts with $\bar{B}_z$ to produce a new component $\bar{\iota}_1$, of the current density, normal to both $\bar{\iota}_0$ and $\bar{B}_z$. This component lies within the XY - plane. The $\bar{\iota}_1$, component, in turn, produces another component, $\bar{\iota}_2$, normal to both $\bar{\iota}_1$ and $\bar{B}_z$, and hence also lies within the XY - plane, but it is rotated $\pi/2$ from $\bar{\iota}_1$. The new component, $\bar{\iota}_2$, produces still another component, $\bar{\iota}_3$, normal to $\bar{\iota}_2$ within the XY - plane. Since $\bar{\iota}_3$ has been rotated π from $\bar{\iota}_1$, it is in the same direction as $\bar{\iota}_1$, but of the opposite sense. Also, the component $\bar{\iota}_4$, produced by $\bar{\iota}_3$, and is also of the opposite sense. Continuing, an infinite alternating series is obtained for the two directions lying within the XY - plane. These series will converge under certain conditions. They will be obtained analytically in the following analysis.

In the presence of the geodesic field, the force equation (11), for each electron, must be modified in accordance with ampere's force law to become,

$$m \frac{d\bar{V}}{dt} + \nu m \bar{V} + e \bar{V} \times \bar{B} = -e\bar{E} . \tag{36}$$

Equation (36) is to be solved by iteration. That is, the solution $\bar{V}_\circ$ of equation (12) will be substituted into $e \bar{V} \times \bar{B}$ and a correction term $\bar{V}_1$, obtained. The $\bar{V}_1$ term will then be substituted into $e \bar{V} \times \bar{B}$ to obtain another correction term $\bar{V}_2$, etc. Before substitution, the successive cross products will be formulated. From table (35),

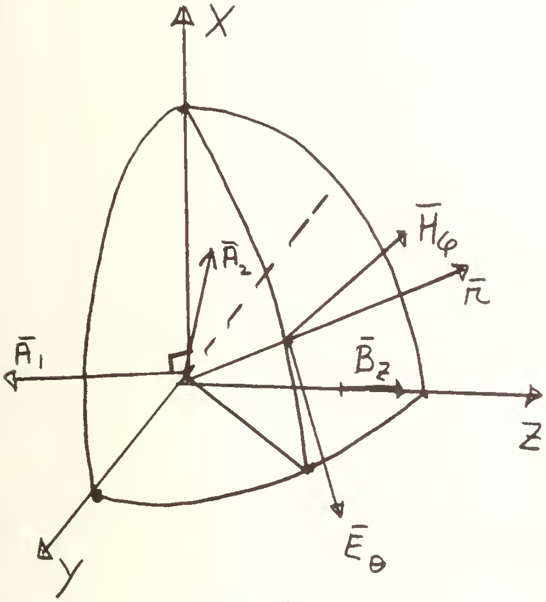


Figure 5

$$\bar{a}_\theta = -\bar{a}_x \sin\theta + \bar{a}_y \cos\theta \cos\varphi + \bar{a}_z \cos\theta \sin\varphi, \text{ and} \tag{37}$$

$$\bar{A}_1 = \bar{a}_\theta \times \bar{a}_z = \bar{a}_1 \sin\zeta , \; \zeta = \cos^{-1} (\bar{a}_\theta \cdot \bar{a}_z) ,$$

$$\bar{A}_1 = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ -\sin\theta & \cos\theta\cos\varphi & \cos\theta\sin\varphi \\ 0 & 0 & 1 \end{vmatrix} = \bar{a}_x \cos\theta\cos\varphi + \bar{a}_y \sin\theta \tag{38}$$

$$\bar{A}_2 = (\bar{a}_\theta \times \bar{a}_z) \times \bar{a}_z = \bar{a}_2 \sin\zeta = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \cos\theta\cos\varphi & \sin\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \bar{a}_x \sin\theta - \bar{a}_y \cos\theta\cos\varphi \tag{39}$$

But from table (35),

$$E_x = -E_\circ \sin\theta , \; E_y = E_\circ \cos\theta\cos\varphi , \; E_z = E_\circ \cos\theta\sin\varphi . \tag{40}$$

Hence,

$$\bar{A}_1 = \bar{a}_x E_y - \bar{a}_y E_x \tag{41}$$

and

$$\bar{A}_2 = -(\bar{a}_x E_x + \bar{a}_y E_y). \tag{42}$$

From equation (38) and (39), it may be seen that both $\bar{A}_1$ and $\bar{A}_2$ lie within the XY - plane, and that their slopes are negative reciprocals. That is to say, $\bar{A}_2$ is rotated $\pi/2$ from $\bar{A}_1$. Furthermore, since each unit vector $\bar{a}_i$, j = 1, 2, ---, is within the XY - plane, each $\bar{a}_{i+1} = \bar{a}_i \times \bar{a}_z$ also is

within the XY - plane, and $\bar{a}_{i+1}$ is in space quadrature with $\bar{a}_i$. Incidentally, the unit vectors $\bar{a}_2$, $\bar{a}_z$ and $\bar{a}_\theta$ are also coplanar, with $\bar{a}_2$ at an angle $\pi/2 + \zeta$ with $\bar{a}_\theta$.

For the iteration, from equations (12) and (10),

$$\bar{V}_\circ = \frac{-eE_\circ \bar{a}_\theta}{jm(\omega - j\nu)}, \quad \bar{F}_1 = -e\bar{V}_\circ \times \bar{B}_z = jm(\omega - j\nu)\bar{V}_1. \qquad (43)$$

Thus, upon defining the gyro frequency,

$$\omega_b = \frac{eB_z}{m} \qquad (44)$$

$$\bar{V}_1 = \frac{e^2 \bar{E}_{\circ\theta} \times \bar{B}_z}{[jm(\omega - j\nu)]^2} = \frac{-eE_\circ \omega_b \bar{A}_1}{m(\omega - j\nu)^2} =$$

$$\frac{-e\omega_b E_\circ \sin\zeta}{m(\omega - j\nu)^2}\bar{a}_1 . \qquad (45)$$

Repeating the procedure for

$$F_2 = -e\bar{V}_1 \times B_z = jm(\omega - j\nu)\bar{V}_2 , \qquad (46)$$

$$\bar{V}_2 = \frac{e^2 E_\circ B_z \omega_b \bar{A}_1 \times \bar{a}_z}{jm^2(\omega - j\nu)^3} =$$

$$\frac{-je\omega_b^2 E_\circ \sin\zeta}{m^2(\omega - j\nu)^3}\bar{a}_2 \qquad (47)$$

For

$$\bar{F}_3 = -e\bar{V}_2 \times \bar{B}_z = jm(\omega - j\nu)\bar{V}_3 , \qquad (48)$$

$$\bar{V}_3 = \frac{e\omega_b^3 E_\circ \sin\zeta}{m(\omega - j\nu)^4}\bar{a}_3 , \quad \bar{a}_3 = -\bar{a}_1 . \qquad (49)$$

Continuing,

$$\bar{V}_4 = \frac{je\omega_b^4 E_\circ \sin\zeta}{m(\omega - j\nu)^5}\bar{a}_4 , \quad \bar{a}_4 = -\bar{a}_2 , \qquad (50)$$

$$\bar{V}_5 = \frac{-e\omega_b^5 E_\circ \sin\zeta}{m(\omega - j\nu)^6}\bar{a}_5 , \quad \bar{a}_5 = \bar{a}_1 , \qquad (51)$$

$$\bar{V}_6 = \frac{-je\omega_b^6 E_\circ \sin\zeta}{m(\omega - j\nu)^7}\bar{a}_6 , \quad \bar{a}_6 = \bar{a}_2 . \qquad (52)$$

The above procedure may be continued indefinitely.

Upon summing the corrective components (43) through (52) ad infinitum, factoring $\frac{-j\,\omega\,e_0}{Ne}$, and substituting ω_p, the velocity vector for an electron becomes,

$$\bar{V} = \frac{+j\,\omega\epsilon_0 E_0}{Ne}\left\{\bar{a}_\theta \frac{\omega_p^{\,2}}{\omega(\omega-j\nu)} + j\,\bar{a}_1\left[\frac{\omega_p^{\,2}\,\omega_b}{\omega(\omega-j\nu)^2} + \frac{\omega_p^{\,2}\,\omega_b^{\,3}}{\omega(\omega-j\nu)^4} + \frac{\omega_p^{\,2}\,\omega_b^{\,5}}{\omega(\omega-j\nu)^6} + ---\right]\sin\zeta - \bar{a}_2\left[\frac{\omega_p^{\,2}\,\omega_b^{\,2}}{\omega(\omega-j\nu)^3} + \frac{\omega_p^{\,2}\,\omega_b^{\,4}}{\omega(\omega-j\nu)^5} + \frac{\omega_p^{\,2}\,\omega_b^{\,6}}{\omega(\omega-j\nu)^7} + ---\right]\sin\zeta\right\} \tag{53}$$

or,

$$\bar{V} = \frac{+j\,\omega\epsilon_0 E_0}{Ne}\left\{\bar{a}_\theta \frac{\omega_p^{\,2}}{\omega(\omega-j\nu)} + j\,\bar{a}_1 \frac{\omega_p^{\,2}\,\omega_b}{\omega(\omega-j\nu)^2}\left[1 + \frac{\omega_b^{\,2}}{(\omega-j\nu)^2} + \frac{\omega_b^{\,4}}{(\omega-j\nu)^4} + ---\right]\sin\zeta - \bar{a}_2 \frac{\omega_p^{\,2}\,\omega_b^{\,2}}{\omega(\omega-j\nu)^3}\left[1 + \frac{\omega_b^{\,2}}{(\omega-j\nu)^2} + \frac{\omega_b^{\,4}}{(\omega-j\nu)^4} + ---\right]\sin\zeta\right\}. \tag{54}$$

The infinite series within the brackets of equation (54) converges for

$$\frac{\omega_b^2}{\omega^2+\nu^2} < 1\ , \tag{55}$$

that is, for

$$\omega > \omega_b\sqrt{1-(\nu/\omega_b)^2}\ . \tag{56}$$

Hence, for the frequency range of convergence, the brackets become,

$$\left[1 + \frac{\omega_b^{\,2}}{(\omega-j\nu)^2} + ---\right] = \frac{1}{1-\omega_b^{\,2}/(\omega-j\nu)^2} = \frac{(\omega-j\nu)^2}{(\omega-j\nu)^2-\omega_b^{\,2}}\ , \tag{57}$$

and hence,

$$\bar{V} = \frac{+j\,\omega\epsilon_0 E_0}{Ne}\left\{\bar{a}_\theta \frac{\omega_p^{\,2}}{\omega(\omega-j\nu)} + \frac{\omega_b\,\omega_p^{\,2}\sin\zeta}{\omega[(\omega-j\nu)^2-\omega_b^{\,2}]} \times \left[j\,\bar{a}_1 - \bar{a}_2\frac{\omega_b}{\omega-j\nu}\right]\right\} \tag{58}$$

3.2 Path of An Electron

For an examination of the path followed by an electron, it is preferable to eliminate ω_p^2 in equation (58), and integrate in time to obtain a radius vector, $\bar{r}$, whose terminus describes the path. Accordingly,

$$\bar{r} = E_o \frac{e}{m} \left\{ \bar{a}_\theta \frac{1}{\omega(\omega - j\nu)} + \frac{\omega_b \sqrt{1 - \cos^2 \theta \sin^2 \varphi}}{\omega \left[(\omega - j\nu)^2 - \omega_b^2 \right]} \left[j \bar{a}_1 - \bar{a}_2 \frac{\omega_b}{\omega - j\nu} \right] \right\} , \tag{59}$$

in which, from equation (37), $\sin \zeta$ has been substituted in the form

$$\sin \zeta = \sqrt{1 - \cos^2 \theta \sin^2 \varphi} \ . \tag{60}$$

Assuming $\nu \approx 0$, equation (59) shows that as the electron attempts to vibrate in a path parallel with the exciting field, it has an additional elliptical component within a plane normal to the geodesic field, and resonates at $\omega = \omega_b$. For frequencies $\omega < \omega_b$, the ionosphere acts as a conducting medium, for the current density produced by the magnetic field, rather than as a dielectric medium.

The electron path, as it spirals about the direction of the exciting electric field, is somewhat like a trochoidal epicycloid. The electron current density, $\bar{\iota} = - N e \bar{V}$, may be considered as a source of radiation, a vector potential formulated, and the resulting fields computed therefrom in accordance with Huygen's principle. Thus, the electric fields are no longer confined to the path of the driving field. It will be shown later that $\bar{D}$, $\bar{H}$, and $\bar{\gamma}$ constitute an orthogonal system, with $\bar{E}$ not necessarily orthogonal to $\bar{\gamma}$.

3.3 The Curl $\bar{H}$ Equation

The current density will be formulated from equation (58), by multiplication with $- Ne$, and substituted into curl $\bar{H}$ in order to formulate the dielectric tensor. The dielectric tensor subsequently will be used to determine the indices of refraction. The resulting curl equation is,

$$\nabla \times \bar{H} = j\,\omega\epsilon_0 E_0 \left\{ \bar{a}_\theta \left[1 - \frac{\omega_p^2}{\omega(\omega - j\nu)}\right] - j\,\bar{a}_1 \frac{\omega_b\,\omega_p^2 \sin\zeta}{\omega[(\omega - j\nu)^2 - \omega_b^2]} + \bar{a}_2 \frac{\omega_b^2\,\omega_p^2 \sin\zeta}{\omega(\omega - j\nu)[(\omega - j\nu)^2 - \omega_b^2]} \right\} \tag{61}$$

The wave corresponding to the first term of the right member of equation (61) is sometimes called the ordinary wave, and the wave corresponding to the second and third terms is companionably called the extraordinary wave.

3.31 Cartesian Coordinates

Using $\bar{E}_\theta = \bar{a}_x E_x + \bar{a}_y E_y + \bar{a}_z E_z$ along with equations (41) and (42), equation (61) may be re-written,

$$\begin{aligned}
\frac{1}{j\,\omega\epsilon_0} \nabla \times \bar{H} = {} & \bar{a}_x \left\{ E_x \left[1 - \frac{\omega_p^2}{\omega(\omega - j\nu)}\right] - jE_y \frac{\omega_b\,\omega_p^2}{\omega\left[(\omega - j\nu)^2 - \omega_b^2\right]} \right. \\
& \left. - E_x \frac{\omega_b^2\,\omega_p^2}{\omega(\omega - j\nu)[(\omega - j\nu)^2 - \omega_b^2]} \right\} + \bar{a}_y \left\{ E_y \left[1 - \frac{\omega_p^2}{\omega(\omega - j\nu)}\right] + \right. \\
& \left. j\,E_x \frac{\omega_b\,\omega_p^2}{\omega[(\omega - j\nu)^2 - \omega_b^2]} - E_y \frac{\omega_b^2\,\omega_p^2}{\omega(\omega - j\nu)[(\omega - j\nu)^2 - \omega_b^2]} \right\} \\
& + \bar{a}_z E_z \left[1 - \frac{\omega_p^2}{\omega(\omega - j\nu)}\right]
\end{aligned} \tag{62}$$

Now let

$$\begin{aligned}
\tilde{X} &= 1 - \frac{\omega_p^2}{\omega(\omega - j\nu)} - \frac{\omega_b^2\,\omega_p^2}{\omega(\omega - j\nu)[(\omega - j\nu)^2 - \omega_b^2]} = \\
& 1 - \frac{\omega_p^2(\omega - j\nu)}{\omega[(\omega - j\nu)^2 - \omega_b^2]},
\end{aligned} \tag{63}$$

$$\tilde{Y} = \frac{\omega_b\,\omega_p^2}{\omega[(\omega - j\nu)^2 - \omega_b^2]}, \quad \tilde{Z} = 1 - \frac{\omega_p^2}{\omega(\omega - j\nu)}, \tag{64}$$

and substitute into (62) to obtain,

$$\frac{1}{j\,\omega\epsilon_0} \nabla \times \bar{H} = \bar{a}_x (E_x \tilde{X} - j\,E_y \tilde{Y}) + \bar{a}_y (E_y \tilde{X} + j\,E_x \tilde{Y}) + \bar{a}_z E_z \tilde{Z}\,. \tag{65}$$

For cross reference to literature employing the standard URSI symbols,

$$X = \frac{\omega_p^2}{\omega^2}, \quad Y = \frac{\omega_b}{\omega}, \quad \Gamma = 1 - j\ \nu/\omega, \tag{66}$$

the symbols in (63) and (64) are equivalent to,

$$\widetilde{X} = 1 - \frac{X\,\Gamma}{\Gamma^2 - Y^2}, \quad \widetilde{Y} = \frac{X\ Y}{\Gamma^2 - Y^2}, \quad \widetilde{Z} = 1 - \frac{X}{\Gamma}. \tag{67}$$

The symbols (67) will be used later, but for the present it is more convenient to retain the symbols (63) and (64).

3.32 Rotating Coordinates

The $\bar{a}_x$ and $\bar{a}_y$ components of equation (65) will be broken into left and right hand circular components. Re-arranging equation (65),

$$\frac{1}{j\,\omega\epsilon_0} \nabla \times \bar{H} = \bar{a}_x (E_x \widetilde{X} - j E_y \widetilde{Y}) + j \bar{a}_y (E_x \widetilde{Y} - j E_y \widetilde{X}) + \bar{a}_z E_z \widetilde{Z}, \tag{68}$$

or,

$$\frac{1}{j\,\omega\epsilon_0} \nabla \times \bar{H} = E_x (\bar{a}_x \widetilde{X} + j \bar{a}_y \widetilde{Y}) - j E_y (\bar{a}_x \widetilde{Y} + j \bar{a}_y \widetilde{X}). \tag{69}$$

Now let,

$$\bar{a}_x \widetilde{X} + j \bar{a}_y \widetilde{Y} = A_1 (\bar{a}_x + j \bar{a}_y) + A_2 (\bar{a}_x - j \bar{a}_y), \tag{70}$$

and solve for the undetermined constants A_1 and A_2,

$$\begin{aligned} A_1 + A_2 &= \widetilde{X} & \qquad A_1 &= \frac{\widetilde{X} + \widetilde{Y}}{2} \\ A_1 - A_2 &= \widetilde{Y} & \qquad A_2 &= \frac{\widetilde{X} - \widetilde{Y}}{2} \end{aligned} \tag{71}$$

also let,

$$\bar{a}_x \widetilde{Y} + j \bar{a}_y \widetilde{X} = B_1 (\bar{a}_x + j \bar{a}_y) + B_2 (\bar{a}_x - j \bar{a}_y), \tag{72}$$

and obtain,

$$\begin{aligned} B_1 + B_2 &= \widetilde{Y} & \qquad B_1 &= \frac{\widetilde{X} + \widetilde{Y}}{2} = A_1 \\ B_1 - B_2 &= \widetilde{X} & \qquad B_2 &= \frac{\widetilde{Y} - \widetilde{X}}{2} = -A_2 \end{aligned} \tag{73}$$

Substituting from (71) and (73) into equation (69),

$$\frac{1}{j\,\omega\epsilon_0} \nabla \times \bar{H} = (E_x - j E_y) \frac{\widetilde{X} + \widetilde{Y}}{2} (\bar{a}_x + j \bar{a}_y) + (E_x + j E_y) \times$$

$$\frac{\widetilde{X} - \widetilde{Y}}{2} (\bar{a}_x - j \bar{a}_y) + \bar{a}_z E_z \widetilde{Z}. \tag{74}$$

From equations (63) and (64),

$$\tilde{X} \pm \tilde{Y} = 1 - \frac{\omega_p^2 (\omega - j\nu) \mp \omega_b \omega_p^2}{\omega[(\omega - j\nu)^2 - \omega_b^2]} = 1 - \frac{\omega_p^2 [(\omega - j\nu) \mp \omega_b]}{\omega[(\omega - j\nu) + \omega_b][(\omega - j\nu) - \omega_b]} = 1 - \frac{X}{\Gamma \pm Y} . \qquad (75)$$

upon letting,

$$E_{xy} = 1/2 \sqrt{E_x^2 + E_y^2} , \quad \phi_{xy} = \tan^{-1} \frac{E_y}{E_x} , \qquad (76)$$

equation (74) may be written,

$$\nabla \times \bar{H} = j \omega \epsilon_o \left\{ E_{xy} e^{-j \phi_{xy}} \left(1 - \frac{X}{\Gamma + Y}\right) (\bar{a}_x + j \bar{a}_y) + E_{xy} e^{j \phi_{xy}} \left(1 - \frac{X}{\Gamma - Y}\right) (\bar{a}_x - j \bar{a}_y) + \bar{a}_z E_z \left(1 - \frac{X}{\Gamma}\right) \right\} . \qquad (77)$$

3.321 The Complex Dielectric Tensor

Designating the left hand rotating unit vector by $\bar{a}_L$ and the right hand vector by $\bar{a}_R$, that is,

$$\bar{a}_L = \bar{a}_x + j \bar{a}_y , \quad \bar{a}_R = \bar{a}_x - j \bar{a}_y , \qquad (78)$$

and letting,

$$\epsilon_L' = 1 - \frac{X}{\Gamma + Y} , \quad \epsilon_R' = 1 - \frac{X}{\Gamma - Y} , \quad \epsilon_o' = 1 - \frac{X}{\Gamma} , \qquad (79)$$

equation (77) becomes,

$$\nabla \times \bar{H} = j \omega \epsilon_o \begin{bmatrix} \epsilon_L' & 0 & 0 \\ 0 & \epsilon_R' & 0 \\ 0 & 0 & \epsilon_o' \end{bmatrix} \cdot \begin{bmatrix} E_{xy} & e^{-j \phi_{xy}} \\ E_{xy} & e^{j \phi_{xy}} \\ E_z & \end{bmatrix} \cdot \begin{bmatrix} \bar{a}_L \\ \bar{a}_R \\ \bar{a}_z \end{bmatrix} = j\omega \bar{D} . \qquad (80)$$

3.322 The Complex Indices of Refraction

Equation (80) yields three characteristic waves propagating in a homogeneous anisotropic ionosphere. There are two circularly polarized waves within the XY - plane rotating in opposite senses, and a linear wave parallel to the Z-axis. These waves have three distinct indices of refraction.

Rationalizing the denominators of equation (79),

$$\epsilon_o' = \left[1 - \frac{\omega_p^2}{\omega^2 + \nu^2}\right] - j\left[\frac{\nu\ \omega_p^2}{\omega(\omega^2 + \nu^2)}\right], \qquad (81)$$

$$\epsilon_L' = \left\{1 - \frac{\omega_p^2(\omega + \omega_b)}{\omega[(\omega + \omega_b)^2 + \nu^2]}\right\} - j\left\{\frac{\nu\ \omega_p^2}{\omega[(\omega + \omega_b)^2 + \nu^2]}\right\}, \qquad (82)$$

$$\epsilon_R' = \left\{1 - \frac{\omega_p^2(\omega - \omega_b)}{\omega[(\omega - \omega_b)^2 + \nu^2]}\right\} - j\left\{\frac{\nu\ \omega_p^2}{\omega[(\omega - \omega_b)^2 + \nu^2]}\right\}. \qquad (83)$$

The corresponding indices of refraction will be obtained by extracting the square roots of equations (81), (82) and (83). The ordinary index of refraction, n_o, is identical with that of equation (27). Upon applying equations (26) to equations (81) and (82), the left and right hand indices become,

$$n\Big]_{\substack{L\\R}} = \frac{1}{\sqrt{2}}\left[\left\{1 - \frac{\omega_p^2(\omega \pm \omega_b)}{\omega[(\omega \pm \omega_b)^2 + \nu^2]}\right\} + \left(\left\{1 - \frac{\omega_p^2(\omega \pm \omega_b)^2}{\omega[(\omega \pm \omega_b)^2 + \nu^2]}\right\}^2 + \left\{\frac{\omega_p^2\ \nu^2}{\omega[(\omega \pm \omega_b)^2 + \nu^2]}\right\}^2\right)^{1/2}\right]^{1/2} - j\frac{1}{\sqrt{2}}\left[-\left\{1 - \frac{\omega_p^2(\omega \pm \omega_b)}{\omega[(\omega \pm \omega_b)^2 + \nu^2]}\right\} + \left(\left\{1 - \frac{\omega_p^2(\omega \pm \omega_b^2}{\omega[(\omega \pm \omega_b)^2 + \nu^2]}\right\}^2 + \left\{\frac{\omega_p^2\ \nu^2}{\omega[(\omega \pm \omega_b)^2 + \nu^2]}\right\}^2\right)^{1/2}\right]^{1/2} \qquad (84)$$

It should be kept in mind that equation (80) is normalized, and hence indices (84) apply only for propagation of these characteristic waves. For wave combinations other than the characteristic waves, there will be coupling elements within the dielectric tensor, and a dispersion equation will be required for determining the indices. Which of these waves appear depends upon the direction of incidence, as well as upon the frequency.

3.323 Physical Interpretation

The critical angles of reflection, corresponding to the real parts of the indices of refraction in equation (84), may be determined by equation (30). This will be done for a low loss ionosphere, $\nu \approx 0$, and a physical interpretation formulated.

Setting $\nu = 0$ in (84) and squaring, is equivalent to setting $\nu = 0$ in equations (82) and (83). Hence, for applying equation (30), set

$$1 - \frac{\omega_p^2}{\omega(\omega \pm \omega_b)} = \sin^2 \theta_0 \,, \tag{85}$$

or,

$$\omega^2 \pm \omega_b \, \omega - \omega_p^2 \sec^2 \theta_0 = 0 \,. \tag{86}$$

Taking the positive sign in equation (86) for ω_{cL},

$$\omega_{cL} = -\frac{\omega_b}{2} + \sqrt{(\omega_p \sec \theta_0)^2 + \left(\frac{\omega_b}{2}\right)^2} \,, \tag{87}$$

and taking the negative sign for ω_{cR},

$$\omega_{cR} = +\frac{\omega_b}{2} + \sqrt{(\omega_p \sec \theta_0)^2 + \left(\frac{\omega_b}{2}\right)^2} \,. \tag{88}$$

Since ω_{c_0} determined by equation (81) is identical with ω_c determined by equation (33),

$$\omega_{c_0} = \omega_c = \omega_p \sec \theta_0 \,,$$

while,

$$\omega_{cL} = \sqrt{\omega_c^2 + \left(\frac{\omega_b}{2}\right)^2} - \frac{\omega_b}{2} \,,$$

and

$$\omega_{cR} = \sqrt{\omega_c^2 + \left(\frac{\omega_b}{2}\right)^2} + \frac{\omega_b}{2} \,. \tag{89}$$

For frequencies sufficiently high such that $(\omega_c)^2 >> \left(\frac{\omega_b}{2}\right)^2$,

$$\omega_c \left[1 + \left(\frac{\omega_b}{2\,\omega_c}\right)^2\right]^{1/2} = \omega_c \left[1 + \frac{\omega_b^2}{8\,\omega_c^2} + ---\right] = \omega_c + \frac{\omega_b^2}{8\,\omega_c} + --- \,. \tag{90}$$

Hence,

$$\omega_{cL} \approx \omega_c + \frac{\omega_b^2}{8\,\omega_c} - \frac{\omega_b}{2} \approx \omega_c \,, \tag{91}$$

and,

$$\omega_{cR} \approx \omega_c + \frac{\omega_b^2}{8\,\omega_c} + \frac{\omega_b}{2} \approx \omega_c + \frac{\omega_b}{2} \,. \tag{92}$$

Since the three characteristic waves have distinct indices of refraction, they propagate within the ionosphere along different ray paths, and have

different attenuations. The attenuations spoken of above are those due to $\mathcal{I}m\,[n]$. There are other attenuations due to numerous anamolies of the ionosphere. Because of the different critical frequencies, the right hand wave may be reflected at a frequency for which the other waves pass through the ionosphere.

Recalling that the series in equation (54) diverges for frequencies less than the cyclotron frequency (or gyro frequency), it may be concluded that for $\omega < \omega_b$,

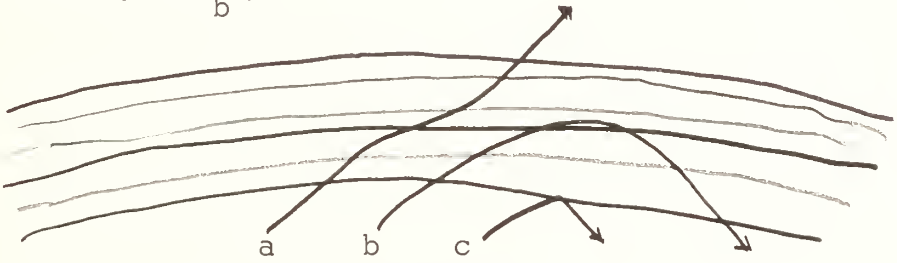


Figure 6

the extraordinary wave confronts a conducting boundary and is reflected as in path c (Fig. 6). For frequencies such that $\omega_b < \omega < \omega_c$, the wave enters the ionosphere, suffers a refraction, and is refracted back to the earth, as in path b (Fig. 6). On the other hand, for frequencies $\omega > \omega_{cR}$, the wave is slightly refracted but passes through the ionosphere as in path a (Fig. 6). The three waves do not necessarily all exist simultaneously, depending upon the angle of the ray path with the geodesic field.

3.4 Azimuth Component of Exciting Field

In the preceding analysis, the exciting field was considered vertically polarized. However, a wave incident upon the ionosphere from a horizontally polarized antenna will have a horizontal component in addition to a vertically polarized component. Hence, it becomes necessary to consider the azimuth component of an exciting field.

Re-writing equation (36) for convenience,

$$\bar{F} = -e\,\bar{E} = m\frac{d\bar{V}}{dt} + \nu\, m\,\bar{V} + e\,\bar{V} \times \bar{B}_z\,, \quad \bar{\iota} = -Ne\bar{V}\,, \tag{36}$$

and considering the exciting field in the form,

$$\bar{E}_\varphi = E_\circ\,(-\bar{a}_x \sin\varphi + \bar{a}_z \cos\varphi)\,, \tag{93}$$

the iteration follows analogously to that for $\bar{E}_\theta$. Accordingly, following steps (43), etc.,

$$\bar{V}_\circ = \frac{-E_\circ e\,\bar{a}_\varphi}{j\,m\,(\omega - j\nu)}\,,\quad \bar{F}_1 = -e\bar{V}_\circ \times \bar{B}_z = j\,m\,(\omega - j\nu)\,\bar{V}_1\,, \tag{94}$$

$$\bar{V}_1 = \frac{e^2 E_\circ\,\bar{a}_\varphi \times \bar{B}_z}{[j\,m(\omega - j\nu)]^2} = \frac{e\,E_\circ\,\omega_b\,(-\bar{a}_y \sin\varphi + \bar{a}_z \cos\varphi) \times \bar{a}_z}{m\,(\omega - j\nu)^2} = \frac{e\,E_\circ\,\omega_b \sin\varphi\,\bar{a}_x}{m\,(\omega - j\nu)^2}\,, \tag{95}$$

$$\bar{V}_2 = -\frac{e^2 E_\circ B_z\,\omega_b \sin\varphi\,\bar{a}_x \times \bar{a}_z}{j\,m^2\,(\omega - j\nu)^3} = \frac{-jeE_\circ\,\omega_b^2 \sin\varphi\,\bar{a}_y}{m\,(\omega - j\nu)^3}\,, \tag{96}$$

$$\bar{V}_3 = \frac{j\,e^2 E_\circ B_z\,\omega_b^2 \sin\varphi\,\bar{a}_y \times \bar{a}_z}{j\,m^2\,(\omega - j\nu)^4} = \frac{eE_\circ\,\omega_b^3 \sin\varphi\,\bar{a}_x}{m\,(\omega - j\nu)^4}\,, \tag{97}$$

$$\bar{V}_4 = \frac{-e^2 E_\circ B_z\,\omega_b^3 \sin\varphi\,\bar{a}_x \times \bar{a}_z}{j\,m^2\,(\omega - j\nu)^5} = \frac{-j\,e\,E_\circ\,\omega_b^4 \sin\varphi\,\bar{a}_y}{m\,(\omega - j\nu)^5} \tag{98}$$

Substituting the above into curl $\bar{H}$, using $\bar{\iota} = -Ne\bar{V}$, and considering $E_\circ\bar{a}_\varphi = \bar{a}_y E_y + \bar{a}_z E_z$, curl $\bar{H}$ =

$$\begin{aligned} j\,\omega\,\epsilon_\circ \Bigg\{ & E_\circ\bar{a}_\varphi \left[1 - \frac{\omega_p^2}{\omega\,(\omega - j\nu)}\right] - j\bar{a}_x E_y \frac{\omega_p^2}{\omega} \left[\frac{\omega_b}{(\omega - j\nu)^2} + \right. \\ & \left. \frac{\omega_b^3}{(\omega - j\nu)^4} + \frac{\omega_b^5}{(\omega - j\nu)^6} + --- \right] - \bar{a}_y E_y \frac{\omega_p^2}{\omega} \left[\frac{\omega_b^2}{(\omega - j\nu)^3} + \right. \\ & \left. \frac{\omega_b^4}{(\omega - j\nu)^5} + \frac{\omega_b^6}{(\omega - j\nu)^7} + --- \right] \Bigg\} \end{aligned} \tag{99}$$

Summing the series over the range of frequencies for which the series converges,

$$\begin{aligned} \text{curl}\ \bar{H} = j\,\omega\,\epsilon_\circ \Bigg\{ & \bar{a}_\varphi E_\circ \left[1 - \frac{\omega_p^2}{\omega(\omega - j\nu)}\right] - j\bar{a}_x E_y \frac{\omega_p^2\,\omega_b}{\omega(\omega - j\nu)^2} \\ & \left[\frac{1}{1 - \dfrac{\omega_b^2}{(\omega - j\nu)^2}}\right] - \bar{a}_y E_y \frac{\omega_p^2\,\omega_b^2}{\omega\,(\omega - j\nu)^3} \left[\frac{1}{1 - \dfrac{\omega_b^2}{(\omega - j\nu)^2}}\right], \end{aligned} \tag{100}$$

or,

$$\text{curl } \bar{H} = j\,\omega\epsilon_0 \left\{ \bar{a}_\varphi E_0 \left[1 - \frac{\omega_p^2}{\omega(\omega - j\nu)} \right] - jE_y \frac{\omega_p^2\ \omega_b}{\omega[(\omega - j\nu)^2 - \omega_b^2]} \left[\bar{a}_x - j\,\bar{a}_y \frac{\omega_b}{\omega - j\nu} \right] \right\}. \tag{101}$$

Substituting for $\bar{a}_\varphi E_0$ of equation (93), and using E_x and E_z,

$$\text{curl } \bar{H} = j\,\omega\epsilon_0 \left[-j\,\bar{a}_x \frac{E_y\ \omega_p^2\ \omega_b}{\omega[(\omega - j\nu)^2 - \omega_b^2]} + \bar{a}_y E_y \left\{ 1 - \frac{\omega_p^2}{\omega(\omega - j\nu)} - \frac{\omega_p^2\ \omega_b^2}{\omega(\omega - j\nu)[(\omega - j\nu)^2 - \omega_b^2]} \right\} + \bar{a}_z E_z \left[1 - \frac{\omega_p^2}{\omega(\omega - j\nu)^2} \right] \right]. \tag{102}$$

Substituting from equations (63) and (64),

$$\text{curl } \bar{H} = j\,\omega\,\epsilon_0 \left\{ -j\,E_y\,(\bar{a}_x \tilde{Y} + j\,\bar{a}_y \tilde{X}) + \bar{a}_z E_z \tilde{Z} \right\}. \tag{103}$$

Let

$$\bar{a}_x \tilde{Y} + j\,\bar{a}_y \tilde{X} = A_1(\bar{a}_x + j\,\bar{a}_y) + A_2(\bar{a}_x - j\,\bar{a}_y), \tag{104}$$

and determine the arbitrary coefficients A_1 and A_2,

$$A_1 + A_2 = \tilde{Y} \qquad A_1 = 1/2\,(\tilde{Y} + \tilde{X})$$

$$A_1 - A_2 = \tilde{X}, \qquad A_2 = 1/2\,(\tilde{Y} - \tilde{X}). \tag{105}$$

Thus, (104) becomes,

$$\frac{1}{j\,\omega\epsilon_0} \text{curl } \bar{H} = -j\,\frac{E_y}{2}(\tilde{X} + \tilde{Y})(\bar{a}_x + j\,\bar{a}_y) + j\,\frac{E_y}{2}(\tilde{X} - \tilde{Y}) \times (\bar{a}_x - j\,\bar{a}_y) + \bar{a}_z E_z \tilde{Z}. \tag{106}$$

Changing to the URSI symbols by substituting from equations (67), (75) and (78),

$$\text{curl } \bar{H} = j\,\omega\epsilon_0 \left\{ -j\,\frac{E_y}{2}\left(1 - \frac{X}{\Gamma + Y}\right)\bar{a}_L + j\,\frac{E_y}{2}\left(1 - \frac{X}{\Gamma - Y}\right)\bar{a}_R + E_z\left(1 - \frac{X}{\Gamma}\right)\bar{a}_z \right\}. \tag{107}$$

Using equations (79),

$$\text{curl } \bar{H} = j\,\omega\,\epsilon_0 \begin{bmatrix} \epsilon_L' & 0 & 0 \\ 0 & \epsilon_R' & 0 \\ 0 & 0 & \epsilon_0' \end{bmatrix} \cdot \begin{bmatrix} -j\ E_y/2 \\ j\ E_y/2 \\ E_z \end{bmatrix} \cdot \begin{bmatrix} \bar{a}_L \\ \bar{a}_R \\ \bar{a}_z \end{bmatrix} = j\,\omega\,\bar{D}. \tag{108}$$

Hence, the same characteristic waves appear as in the case of the vertically polarized exciting wave. Which of these waves appear depends upon the angle of incidence and direction of incidence, as well as upon the frequency.

3.5 Horizontally polarized exciting field

Suppose an exciting electric field is incident upon the ionosphere from a horizontally polarized antenna, the incident field being in the form,

$$\bar{E} = -\frac{E_0}{r} e^{j\omega t - \bar{\gamma} \cdot \bar{r}} \bar{a}_\psi , \tag{109}$$

$-\bar{a}_\psi$ being shown in figure 4. It may be seen that $-\bar{a}_\psi$ is composed of two orthogonal components in the $\bar{a}_\theta$ and $\bar{a}_\varphi$ directions.

If, in equations (40), E_0 is replaced with $E_{0\theta}$, and if in equation (93), E_0 is replaced with $E_{0\varphi}$, then from table (35), E_0 of equation (109) may be expressed by,

$$-E_0 \bar{a}_\psi = \bar{a}_\theta E_{0\theta} + \bar{a}_\varphi E_{0\varphi} \tag{110}$$

with,

$$E_{0\theta} = E_0 \frac{\cos\theta \sin\varphi}{\sqrt{1 - \sin^2\theta \sin^2\varphi}} , \quad E_{0\varphi} = E_0 \frac{\cos\varphi}{\sqrt{1 - \sin^2\theta \sin^2\varphi}} . \tag{111}$$

Also, in equations (40),

$$\sqrt{E_x^2 + E_y^2} = E_0 \sqrt{1 - \cos^2\theta \sin^2\varphi} , \tag{112}$$

and,

$$\tan \phi_{xy} = \frac{E_y}{E_x} = -\cot\theta \cos\varphi . \tag{113}$$

Therefore, for the horizontally polarized exciting electric field vector, equations (77) and (107) may be combined into

$$\begin{aligned} \operatorname{curl} \bar{H} = {} & j\omega\epsilon_0 \left\{ 1/2 \left[e^{-j\phi_{xy}} \epsilon_L' \bar{a}_L + e^{j\phi_{xy}} \epsilon_R' \bar{a}_R \right] \sqrt{1 - \sin^2\theta \cos^2\varphi} + \bar{a}_z \epsilon_0' \cos\theta \sin\varphi \right\} \frac{\cos\theta \sin\varphi \, E_0}{\sqrt{1 - \sin^2\theta \sin^2\varphi}} \\ & + j\omega\epsilon_0 \left\{ j\, 1/2 \left[\epsilon_L' \bar{a}_L - \epsilon_R' \bar{a}_R \right] \sin\varphi + \bar{a}_z \epsilon_0' \cos\varphi \right\} \frac{\cos\varphi \, E_0}{\sqrt{1 - \sin^2\theta \sin^2\varphi}} \end{aligned} \tag{114}$$

To determine the critical frequencies involved in equation (114), it would

seem preferable to consider individually the various normalized components.

4. THE DISPERSION EQUATION

4.1 Maxwell's Equations in Complex Form

The instantaneous Maxwell equations,

$$\operatorname{div} \bar{B} = 0, \ \operatorname{div} \bar{D} = \rho, \ \operatorname{curl} \bar{E} = -\frac{\partial \bar{B}}{\partial t}, \ \operatorname{curl} \bar{H} = \frac{\partial \bar{D}}{\partial t}, \tag{115}$$

$$\bar{\iota} = \sigma \bar{E} = \rho \bar{V} = -Ne\bar{V}, \ \bar{D} = \epsilon' \epsilon_0 \bar{E}, \ \bar{B} = \mu' \mu_0 \bar{H},$$

are possibly more useful for steady state time harmonic cases when considered in the form,

$$\operatorname{div} \bar{B} = 0, \operatorname{div} \bar{D} = 0, \ \operatorname{curl} \bar{E} = -j\omega\mu\bar{H}, \ \operatorname{curl} \bar{H} = \bar{\iota} + j\omega\epsilon\bar{E}. \tag{116}$$

To show that $\operatorname{div} \bar{D} = 0$, first formulate the equation for the continuity of charge by taking $\operatorname{div} \operatorname{curl} \bar{H}$,

$$\operatorname{div} \operatorname{curl} \bar{H} \equiv 0 = \operatorname{div} \bar{\iota} + \frac{\partial}{\partial t} \operatorname{div} \bar{D} = \operatorname{div} \bar{\iota} + \frac{\partial \rho}{\partial t}, \tag{117}$$

or

$$\operatorname{div} \bar{\iota} = -\frac{\partial \rho}{\partial t}. \tag{118}$$

But, for conductors,

$$\operatorname{div} \bar{\iota} = \sigma \operatorname{div} \bar{E} = \frac{\sigma}{\epsilon} \operatorname{div} \bar{D} = \frac{\sigma}{\epsilon} \rho, \tag{119}$$

or,

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon} \rho = 0 \tag{120}$$

The solution of (120) may be written

$$\rho = \rho_0 e^{-(\epsilon/\sigma)t}. \tag{121}$$

From equation (121), a charge density within a lossy medium must vanish with time, and hence must vanish within a steady state condition. Of course, since a dielectric is considered to be free of charges, $\operatorname{div} \bar{D}$ also vanishes within a dielectric.

Now consider the case of a spherical wave front given by,

$$\bar{E} = \frac{\bar{E}_0(\theta, \varphi)}{r} e^{j\omega t - \bar{\gamma} \cdot \bar{r}}, \tag{122}$$

which is sufficiently remote from the source for,

$$\frac{\partial E_0}{\partial \theta} \approx 0, \ \frac{\partial E_0}{\partial \varphi} \approx 0,$$

to hold locally. Also assume $\bar{\gamma} = \bar{a}_r \gamma$. Then,

$$\operatorname{div}\left(\frac{\bar{E}_o}{r} e^{-\bar{\gamma}\cdot\bar{r}}\right) = \frac{1}{r^2 \sin\theta}\left\{\frac{\partial}{\partial r}\left[\frac{\bar{E}_o r^2 \sin\theta}{r} e^{-\bar{\gamma}\cdot\bar{r}} \cdot \bar{a}_r\right]\right\}$$

$$= \bar{a}_r \cdot \frac{\bar{E}_o}{r^2}\left\{-\gamma r\, e^{-\bar{\gamma}\cdot\bar{r}} + e^{-\bar{\gamma}\cdot\bar{r}}\right\} = -\bar{\gamma}\cdot\frac{\bar{E}_o}{r} e^{-\bar{\gamma}\cdot\bar{r}} + \frac{1}{r^2}\,\bar{\gamma}\cdot\bar{E}_o\, e^{-\bar{\gamma}\cdot\bar{r}} \tag{123}$$

But, the wave is assumed to be a radiated wave such that the inverse square terms vanish. Hence, in practice, for a radiated wave,

$$\operatorname{div}\left[\frac{\bar{E}_o(\theta,\varphi)}{r} e^{-\bar{\gamma}\cdot\bar{r}}\right] = -\bar{\gamma}\cdot\frac{\bar{E}_o(\theta,\varphi)}{r} e^{-\bar{\gamma}\cdot\bar{r}} \tag{124}$$

Likewise, consider

$$\operatorname{curl}\left(\frac{\bar{E}_o}{r} e^{-\bar{\gamma}\cdot\bar{r}}\right) = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \bar{a}_r & r\,\bar{a}_\theta & r\sin\theta\,\bar{a}_\varphi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\varphi \\ \frac{E_{o r}}{r} e^{-\bar{\gamma}\cdot\bar{r}} & \frac{r E_{o\theta}}{r} e^{-\bar{\gamma}\cdot\bar{r}} & \frac{r\sin\theta}{r} E_{o\varphi} e^{-\bar{\gamma}\cdot\bar{r}} \end{vmatrix}$$

$$= \bar{a}_r \frac{\cos\theta}{r^2\sin\theta} E_{o\varphi} e^{-\bar{\gamma}\cdot\bar{r}} - \bar{a}_\theta \frac{r\sin\theta}{r^2\sin\theta} E_{o\varphi}\left(-\gamma e^{-\bar{\gamma}\cdot\bar{r}}\right) + \bar{a}_\varphi \frac{E_{o\theta}}{r}\left(-\gamma e^{-\bar{\gamma}\cdot\bar{r}}\right). \tag{125}$$

Again, assuming a radiated field for which the inverse square terms vanish,

$$\operatorname{curl}\left(\frac{E_o}{r} e^{-\bar{\gamma}\cdot\bar{r}}\right) = \bar{a}_r\, 0 - \bar{a}_\theta\left(-\gamma \frac{\bar{E}_{o\varphi}}{r} e^{-\bar{\gamma}\cdot\bar{r}}\right) + \bar{a}_\varphi\left(-\gamma \frac{E_{o\theta}}{r} e^{-\bar{\gamma}\cdot\bar{r}}\right). \tag{126}$$

But,

$$\bar{\gamma} \times \bar{E}_o = \begin{vmatrix} \bar{a}_r & \bar{a}_\theta & \bar{a}_\varphi \\ \gamma & 0 & 0 \\ E_{o r} & E_{o\theta} & E_{o\varphi} \end{vmatrix} = \bar{a}_r\, 0 - \bar{a}_\theta \gamma E_{o\varphi} + \bar{a}_\varphi \gamma E_{o\theta}. \tag{127}$$

Therefore, in practice,

$$\operatorname{curl}\left[\frac{\bar{E}_0(\theta\,,\varphi)}{r}\,e^{-\bar{\gamma}\,.\,\bar{r}}\right] = -\bar{\gamma}\ \mathrm{x}\ \frac{\bar{E}_0(\theta\,,\varphi)}{r}\,e^{-\bar{\gamma}\,.\,\bar{r}}\quad . \tag{128}$$

Also, in case the relative constant is a tensor, $\bar{\bar{\epsilon}}'$, then

$$\bar{D} = \epsilon_0\bar{\bar{\epsilon}}'\ .\ \bar{E}\ . \tag{129}$$

Therefore, the complex form of Maxwell's equations for radiated fields may be written,

$$\bar{\gamma}\ .\ \bar{H} = 0,\ \bar{\gamma}\ .\ \epsilon_0\bar{\bar{\epsilon}}'.\ \bar{E} =, 0,\ \bar{\iota} = -\ Ne\bar{V},\ \bar{\gamma}\ \mathrm{x}\ \bar{E} = j\ \omega\mu_0\ \bar{H},\ -\bar{\gamma}\ \mathrm{x}\ \bar{H} = -Ne\bar{V} + j\ \omega\epsilon_0\,\bar{E} = j\ \omega\ \epsilon_0\bar{\bar{\epsilon}}'\ .\ \bar{E}\ , \tag{130}$$

in which, $\bar{\bar{\epsilon}}'$ is defined by

$$\bar{\bar{\epsilon}}'\ .\ \bar{E} = -\ Ne\bar{V}/\ j\omega\epsilon_0 + \bar{E}\ , \tag{131}$$

The wave equation becomes,

$$\bar{\gamma}\ \mathrm{x}\ (\bar{\gamma}\ \mathrm{x}\ \bar{E}) - k_0^2\ \bar{\bar{\epsilon}}'\ .\ \ \bar{E} = 0\ . \tag{132}$$

4.11 Orientation of field Vectors

From equations (129) and (130), the relative orientation of the field vectors may be obtained. For this purpose, consider

$$\bar{\gamma}\ \ .\ \bar{D} = 0,\ \bar{\gamma}\ .\ \ \bar{H} = 0\ \ , \tag{133}$$

from which it may be concluded that $\bar{\gamma}$ is normal to both $\bar{D}$ and $\bar{H}$. Then consider,

$$-\ \bar{\gamma}\ \ \mathrm{x}\ \bar{H} = +\ j\ \omega\ \bar{D}, \tag{134}$$

from which it appears that $\bar{D}$ is normal to both $\bar{\gamma}$ and $\bar{H}$. Therefore, the three vectors $\bar{D}$, $\bar{H}$, and $\bar{\gamma}$ constitute an orthogonal triple. Finally consider

$$\bar{\gamma}\ \mathrm{x}\ \bar{E} = j\ \omega\ \mu_0\ \ \bar{H}\quad , \tag{135}$$

from which it is seen that $\bar{H}$ is also normal to the plane containing $\bar{\gamma}$ and $\bar{E}$.

Assuming $\bar{\gamma}$ to beat an arbitrary angle ψ with the geodesic field $\bar{B}_0$, the relative orientation of the unit vectors is then illustrated in figure 7.

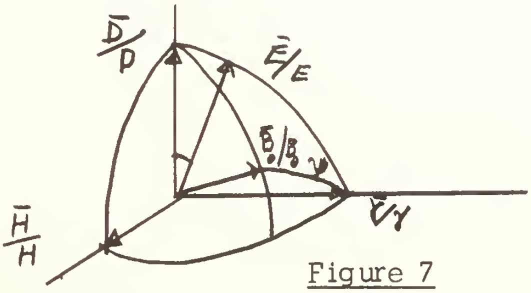


Figure 7

Thus, the TEM incident wave becomes a complex TM wave within the ionosphere.

4.2 The Dispersion Equation

For $\bar{\gamma} = j k_o \bar{n}$, the wave equation reduces to,

$$\bar{n} \times (\bar{n} \times \bar{E}) + \bar{\bar{\epsilon}}' \cdot \bar{E} = 0 . \tag{136}$$

upon expanding the cross products and providing for all components of the dielectric tensor, equation (133) may be expressed as,

$$\begin{bmatrix} -(n_y^2 + n_z^2) & n_x n_y & n_x n_z \\ n_y n_x & -(n_x^2 + n_z^2) & n_y n_z \\ n_z n_x & n_z n_y & -(n_x^2 + n_y^2) \end{bmatrix} \cdot \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} + \begin{bmatrix} \epsilon'_{xx} & \epsilon'_{xy} & \epsilon'_{xz} \\ \epsilon'_{yx} & \epsilon'_{yy} & \epsilon'_{yz} \\ \epsilon'_{zx} & \epsilon'_{zy} & \epsilon'_{zz} \end{bmatrix} \cdot \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0 \tag{137}$$

Combining the tensors,

$$\begin{bmatrix} \epsilon'_{xx} - n_y^2 - n_z^2 & \epsilon'_{xy} + n_x n_y & \epsilon'_{xz} + n_x n_z \\ \epsilon'_{y} + n_y n_x & \epsilon'_{yy} - n_x^2 - n_z^2 & \epsilon'_{yz} + n_y n_z \\ \epsilon'_{zx} + n_z n_x & \epsilon'_{zy} + n_z n_y & \epsilon'_{zz} - n_y^2 - n_x^2 \end{bmatrix} \cdot \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0 . \tag{138}$$

Equations (136) constitute three homogeneous linear equations defining the components of $\bar{E}$. A necessary and sufficient condition that a solution other than the trivial exists, is that the determinant of the coefficient matrix vanishes.. The resulting characteristic equation is called the dispersion equation, as its eigenvalues determine the indices of refraction and coefficients of attenuation. That is, the dispersion equation is,

$$\begin{vmatrix} \epsilon'_{xx} - n_y^2 - n_z^2 & \epsilon'_{xy} + n_x n_y & \epsilon'_{xz} + n_x n_z \\ \epsilon'_{yx} + n_y n_x & \epsilon'_{yy} - n_x^2 - n_z^2 & \epsilon'_{yz} + n_y n_z \\ \epsilon'_{zx} + n_z n_x & \epsilon'_{zy} + n_z n_y & \epsilon'_{zz} - n_y^2 - n_x^2 \end{vmatrix} = 0 . \tag{139}$$

4.21 Special Cases

As an indication of the application of the dispersion equation, some special cases will be considered. For this purpose, re-write equations (68) and (103), respectively, in the following forms. For E_θ,

$$-\bar{\gamma} \times \bar{H} = j\,\omega\epsilon_o \begin{bmatrix} 1-\frac{X\,\Gamma}{\Gamma^2-Y^2} & \frac{-j\,X\,Y}{\Gamma^2-Y^2} & 0 \\ j\,\frac{X\,Y}{\Gamma^2-Y^2} & 1-\frac{X\,\Gamma}{\Gamma^2-Y^2} & 0 \\ 0 & 0 & 1-\frac{X}{\Gamma} \end{bmatrix} \cdot \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}, \tag{140}$$

and for E_φ,

$$-\bar{\gamma} \times \bar{H} = j\,\omega\,\epsilon_o \begin{bmatrix} 0 & -j\,\frac{X\,Y}{\Gamma^2-Y^2} & 0 \\ 0 & 1-\frac{X\,\Gamma}{\Gamma^2-Y^2} & 0 \\ 0 & 0 & 1-\frac{X}{\Gamma} \end{bmatrix} \cdot \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \tag{141}$$

Let

$$\bar{\gamma} = \gamma\bar{a}_r = j\,k_o\bar{n} = j\,k_o n\,(\bar{a}_x\cos\theta + \bar{a}_y\sin\theta\cos\varphi + \bar{a}_z\sin\theta\cos\varphi) \tag{142}$$

Using the dielectric tensor from equation (140) along with the dispersion equation (139), and also substituting from equation (142), for $\phi = \pi/2$,

$$\begin{vmatrix} 1-n^2\sin^2\theta-\frac{\Gamma X}{\Gamma^2-Y^2} & \frac{-j\,X\,Y}{\Gamma^2-X^2} & n^2\sin\theta\cos\theta \\ \frac{j\,X\,Y}{\Gamma^2-Y^2} & 1-n^2-\frac{\Gamma X}{\Gamma^2-Y^2} & 0 \\ n^2\sin\theta\cos\theta & 0 & 1-n^2\cos^2\theta-\frac{X}{\Gamma} \end{vmatrix} = 0 \tag{143}$$

and for $\varphi = 0$,

$$\begin{vmatrix} 1-\frac{\Gamma X}{\Gamma^2-Y^2}-n^2\sin^2\theta & \frac{-j\,X\,Y}{\Gamma^2-X^2}+n^2\sin\theta\cos\theta & 0 \\ \frac{j\,X\,Y}{\Gamma^2-Y^2}+n^2\sin\theta\cos\theta & 1-\frac{\Gamma X}{\Gamma^2-Y^2}-n^2\cos^2\theta & 0 \\ 0 & 0 & 1-\frac{X}{\Gamma}-n^2 \end{vmatrix} = 0 \tag{144}$$

Equation (143) is the dispersion equation for a vertically polarized wave propagating in a longitudinal direction, whereas equation (144) is for a vertically polarized wave propagating transverse to the geodesic field.

Now, in equation (143), let $\theta = \pi/2$,

$$\begin{vmatrix} 1-n^2 - \frac{\Gamma X}{\Gamma^2 - Y^2} & \frac{-jXY}{\Gamma^2 - X^2} & 0 \\ \frac{jXY}{\Gamma^2 - X^2} & 1 - n^2 - \frac{\Gamma X}{\Gamma^2 - X^2} & 0 \\ 0 & 0 & 1 - \frac{\Gamma X}{\Gamma^2 - X^2} \end{vmatrix} = 0 \tag{145}$$

Thus,

$$\left(1 - \frac{\Gamma X}{\Gamma^2 - Y^2} - n^2\right)^2 - \left(\frac{XY}{\Gamma^2 - Y^2}\right)^2 = 0, \tag{146}$$

and

$$n^2 = 1 - \frac{X}{\Gamma \pm Y} . \tag{147}$$

For equation (144) with $\theta = \pi/2$,

$$\begin{vmatrix} 1 - n^2 - \frac{\Gamma X}{\Gamma^2 - Y^2} & \frac{-jXY}{\Gamma^2 - Y^2} & 0 \\ \frac{jXY}{\Gamma^2 - Y^2} & 1 - \frac{\Gamma X}{\Gamma^2 - Y^2} & 0 \\ 0 & 0 & 1 - \frac{X}{\Gamma} - n^2 \end{vmatrix} = 0 . \tag{148}$$

one solution is,

$$n^2 = 1 - \frac{X}{\Gamma} . \tag{149}$$

For the other solution,

$$\left(1 - \frac{\Gamma X}{\Gamma^2 - Y^2} - n^2\right)\left(1 - \frac{\Gamma X}{\Gamma^2 - Y^2}\right) - \left(\frac{XY}{\Gamma^2 - Y^2}\right)^2 = 0, \tag{150}$$

$$n^2 = 1 - \frac{\Gamma X}{\Gamma^2 - Y^2} - \frac{X^2 Y^2}{(\Gamma^2 - \Gamma X - Y^2)(\Gamma^2 - Y^2)} , \tag{151}$$

or,

$$n^2 = 1 - \frac{X}{\Gamma - \frac{Y}{\Gamma - X}} \tag{152}$$

Solution (147) is for a vertically polarized wave in the longitudinal direction whose exciting field is normal to the geodesic field, and solutions (149) and (152) are analogous solutions for the transverse case.

For the azimuth, or horizontal component of the exciting field, the dispersion equation for $\varphi = \pi/2$, that is, for longitudinal propagation, becomes,

$$\begin{vmatrix} -n^2\sin^2\theta & -j\dfrac{XY}{\Gamma^2-Y^2} & n^2\sin\theta\cos\theta \\ 0 & 1-n^2-\dfrac{\Gamma X}{\Gamma^2-X^2} & 0 \\ n^2\sin\theta\cos\theta & 0 & 1-n^2\cos^2\theta-\dfrac{X}{\Gamma} \end{vmatrix} = 0 . \tag{153}$$

Expanding,

$$\frac{jXY}{\Gamma^2-Y^2}(0) + \left(1-\frac{\Gamma X}{\Gamma^2-X^2}-n^2\right)\begin{vmatrix} -\sin\theta & n^2\sin\theta\cos\theta \\ \cos\theta & 1-n^2\cos^2\theta-\dfrac{X}{\Gamma} \end{vmatrix} = 0, \tag{154}$$

from which a solution is obtained,

$$n^2 = 1-\frac{\Gamma X}{\Gamma^2-Y^2} \tag{155}$$

For the other factor,

$$-\left(1-\frac{X}{\Gamma}\right)\sin\theta + n^2\sin\theta\cos^2\theta - n^2\sin\theta\cos^2\theta = -\left(1-\frac{X}{\Gamma}\sin\theta\right),$$

which is constant. Therefore, solution (155) is the sole solution for this case.

For transverse propagation, $\varphi = 0$, and the exciting field is parallel with the geodesic field. Hence, in this case, the ordinary solution holds, namely

$$n^2 = 1-\frac{X}{\Gamma} . \tag{156}$$

5. APPLETON EQUATION FOR ARBITRARILY ORIENTED GEODESIC FIELD

5.1 The Appleton Euqation

The Appleton equation, sometimes referred to as the Appleton-Hartree equation, is an equation for determining the complex index of refraction in

a homogeneous, anisotropic ionosphere. It is customarily derived by choosing one of two coordinate planes as being determined by the $\bar{\gamma}$ and $\bar{B}_o$ vectors. It will be derived herein with an arbitrarily oriented geodesic field so that a greater leeway in the selection of the coordinate axes is permissible for applications. Refer to figure 8.

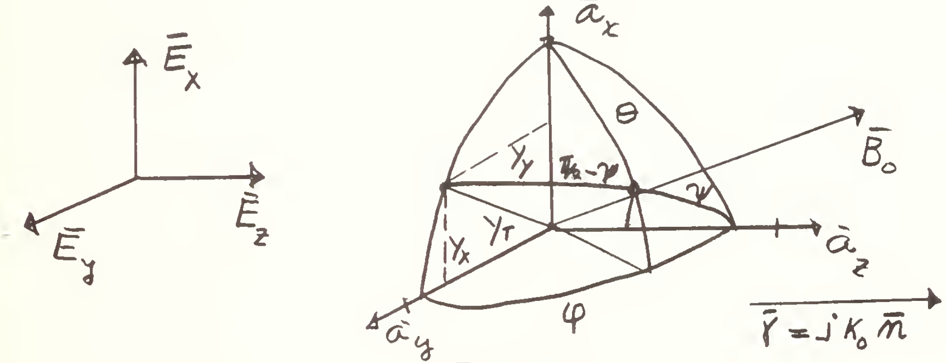


Figure 8

The following symbols will be used,

$$\omega_b = \frac{e B_o}{m}, \; \omega_p^2 = \frac{Ne^2}{m \epsilon_o}, \; \Gamma = 1 - j \nu/\omega, \; X = \frac{\omega_p^2}{\omega^2}, \; Y = \frac{\omega_b}{\omega} = \sqrt{Y_X^2 + Y_y^2 + Y_z^2}, \tag{157}$$

$$Y_X = Y \cos\theta, \; Y_y = Y \sin\theta \cos\varphi, \; Y_z = Y \sin\theta \sin\varphi = Y \cos\psi = Y_L, \tag{158}$$

$$Y_T = Y \sin\psi = Y \sqrt{1 - \sin^2\theta \sin^2\varphi}, \; \frac{e B_o}{j \omega m} = -jy. \tag{159}$$

The geodesic field is taken as,

$$\bar{B}_o = B_o (\bar{a}_X \cos\theta + \bar{a}_y \sin\theta \cos\varphi + \bar{a}_z \sin\theta \sin\varphi), \tag{160}$$

and $\bar{\gamma}$ as,

$$\bar{\gamma} = \bar{a}_z \gamma = j k_o \bar{n}. \tag{161}$$

5.2 Lorentz Conductivity Tensor for An Exciting Wave

The exciting field will be postulated to vary as $\frac{1}{r} e^{j\omega t - \bar{\gamma} \cdot \bar{r}}$.

Hence, Maxwell's equations may be written in the form,

$$\bar{\gamma} \cdot \bar{H} = 0, \; \bar{\gamma} \cdot \epsilon_o \overline{\overline{\epsilon}}' \cdot \bar{E} = 0, \; \bar{\gamma} \times \bar{E} = j\omega\mu_o \vec{H}, \; -\bar{\gamma} \times \bar{H} = \bar{\iota} + j \omega \epsilon_o \bar{E}, \; \bar{\iota} = -N e \bar{V}. \tag{162}$$

Solving for $\bar{\iota}$,

$$\bar{\iota} = -\bar{\gamma} \times \bar{H} - j\omega\epsilon_0 \bar{E} = -\left[j\,\omega\epsilon_0 \bar{E} + \frac{1}{j\,\omega\mu_0}\,\bar{\gamma} \times (\bar{\gamma} \times \bar{E}) \right] = -j\omega\epsilon_0 \times$$

$$\left[\bar{E} + \frac{-k_0^2}{-k_0^2}\,\bar{n} \times (\bar{n} \times \bar{E}) \right]. \tag{163}$$

Now,

$$\bar{n} \times (\bar{n} \times \bar{E}) = \bar{n} \times \left[\bar{n} \times (\bar{a}_x E_x + \bar{a}_y E_y + \bar{a}_z E_z) \right] = n^2 a_z \times (\bar{a}_y E_x -$$

$$\bar{a}_x E_y), \tag{164}$$

or,

$$\bar{n} \times (\bar{n} \times \bar{E}) = -n^2 (\bar{a}_x E_x + \bar{a}_y E_y + \bar{a}_z 0). \tag{165}$$

Therefore,

$$\bar{\iota} = -j\,\omega\epsilon_0 \left[\bar{E} + \bar{n} \times (\bar{n} \times \bar{E}) \right] = -j\,\omega\epsilon_0 \left[\bar{E} - n^2 (\bar{a}_x E_x + \bar{a}_y E_y + \bar{a}_z 0) \right], \tag{166}$$

and hence,

$$\bar{\iota} = j\,\omega\epsilon_0 \begin{bmatrix} n^2 - 1 & 0 & 0 \\ 0 & n^2 - 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}. \tag{167}$$

The Lorentz conductivity, $\bar{\bar{\sigma}}$, is defined by,

$$\bar{\iota} = \bar{\bar{\sigma}} \cdot \bar{E}. \tag{168}$$

Hence, if M^2 is defined by $M^2 = n^2 - 1$, the Lorentz conductivity tensor becomes,

$$\bar{\bar{\sigma}} = j\,\omega\epsilon_0 \begin{bmatrix} n^2 - 1 & 0 & 0 \\ 0 & n^2 - 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = j\,\omega\epsilon_0 \begin{bmatrix} M^2 & 0 & 0 \\ 0 & M^2 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \frac{j\,\omega\epsilon_0}{M^2} \times$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{-1}{M^2} \end{bmatrix}. \tag{169}$$

A corresponding Lorentz resistivity tensor, $\bar{\bar{\rho}}$, may be defined by,

$$\bar{\bar{\rho}} \cdot \bar{\iota} = \bar{\bar{\rho}} \cdot \bar{\bar{\sigma}} \cdot \bar{E} = \bar{\bar{I}} \cdot \bar{E} = \bar{E}, \tag{170}$$

with $\bar{\bar{I}}$ being the identity tensor.

The resistivity tensor may be found from equation (36), which will be repeated here for convenience,

$$m \frac{d\bar{V}}{dt} + \nu m \bar{V} + e \bar{V} \times \bar{B}_o = -e\bar{E}. \tag{36}$$

In the steady state phasor form, equation (36) is,

$$-e\bar{E} = (j\omega m + \nu m)\bar{V} + e\bar{V} \times \bar{B}_o . \tag{171}$$

Using equation (160) for $\bar{B}_o$ in $\bar{V} \times \bar{B}_o$,

$$\bar{V} \times \bar{B}_o = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ V_x & V_y & V_z \\ \cos\theta & \sin\theta\cos\varphi & \sin\theta\sin\varphi \end{vmatrix} , \tag{172}$$

or,

$$\bar{V} \times \bar{B}_o = B_o[\bar{a}_x(V_y \sin\theta\sin\varphi - V_z \sin\theta\cos\varphi) + \bar{a}_y(V_z\cos\theta - V_x \sin\theta\sin\varphi) + \bar{a}_z(V_x \sin\theta\cos\varphi - V_y\cos\theta)]. \tag{173}$$

Substituting equation (173) into equation (171), factoring $j\omega m$ from the first two terms of (171) and using the symbol Γ, the resulting equation is,

$$-eE = j\omega m\,\Gamma(\bar{a}_x V_x + \bar{a}_y V_y + \bar{a}_z V_z) + eB_o[\bar{a}_x(V_y \sin\theta\sin\varphi - V_z \sin\theta\cos\varphi) + \bar{a}_y(V_z\cos\theta - V_x\sin\theta\sin\varphi) + \bar{a}_z(V_x\sin\theta\cos\varphi - V_y\cos\theta)].$$

Substituting symbols (157) into the re-arrangement of equation (174) as a tensor equation, the equation may be written,

$$\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \frac{-j\omega m}{e} \begin{bmatrix} \Gamma & -jY_z & jY_y \\ jY_z & \Gamma & -jY_x \\ -jY_y & jY_x & \Gamma \end{bmatrix} \cdot \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = \frac{-1}{j\omega\epsilon_o X} \times \begin{bmatrix} \Gamma & -jY_z & jY_y \\ jY_z & \Gamma & -jY_x \\ -jY_y & jY_x & \Gamma \end{bmatrix} \cdot \begin{bmatrix} \iota_x \\ \iota_y \\ \iota_z \end{bmatrix} , \tag{175}$$

with the velocity matrix converted to the current density matrix by

multiplying and dividing by - Ne.

From equations (170) and (175), the resistivity tensor may be taken to be,

$$\overline{\overline{\rho}} = - \frac{1}{j\,\omega\epsilon_0\, X} \begin{bmatrix} \Gamma & -jY_z & jY_y \\ jY_z & \Gamma & -jY_x \\ -jY_y & jY_x & \Gamma \end{bmatrix} . \qquad (176)$$

Consider equation (170), and into it, substitute from equation (168). The result is,

$$\bar{E} = \overline{\overline{\rho}} \cdot \bar{\iota} = \overline{\overline{\rho}} \cdot \overline{\overline{\sigma}} \cdot \bar{E} \;,\; \overline{\overline{I}} \cdot \bar{E} - \overline{\overline{\rho}} \cdot \overline{\overline{\sigma}} \cdot \bar{E} = 0, \qquad (177)$$

or,

$$[\, \overline{\overline{\rho}} \cdot \overline{\overline{\sigma}} - \overline{\overline{I}}\,] \cdot \bar{E} = 0 \quad . \qquad (178)$$

Equations (178) constitute a set of the linearly homogeneous equations for the three components of $\bar{E}$. Hence, for a non-trivial solution to exist,

$$\left| \overline{\overline{\rho}} \cdot \overline{\overline{\sigma}} - \overline{\overline{I}} \right| = 0 \quad . \qquad (179)$$

Multiplying (169) by (176) and substituting into equation (179), the characteristic equation becomes,

$$- \frac{M^6}{X^3} \begin{vmatrix} \Gamma + \frac{X}{M^2} & -jY_z & \frac{jY_y}{M^2} \\ jY_z & \Gamma + \frac{X}{M^2} & \frac{-jY_x}{M^2} \\ -jY_y & jY_x & \frac{\Gamma - X}{M^2} \end{vmatrix} = 0 \quad . \qquad (180)$$

Expanding the determinant,

$$(\Gamma + \frac{X}{M^2})^2 (\Gamma - X) + jY_x Y_y Y_z - jY_x Y_y Y_z - Y_y^2 (\Gamma + \frac{X}{M^2}) - Y_x^2 (\Gamma + \frac{X}{M^2}) - Y_z^2 (\Gamma - X) = 0, \qquad (181)$$

or,

$$(\Gamma + \frac{X}{M^2})^2 - \frac{Y_T^2}{\Gamma - X} (\Gamma + \frac{X}{M}) - Y_L^2 = 0, \qquad (182)$$

in which substitutions $Y_L = Y_z$ and $Y_T^2 = Y_x^2 + Y_y^2$ were made.

Solving for $\Gamma + \frac{X}{M}$,

$$\Gamma + \frac{X}{M^2} = 1/2 \left[\frac{Y_T^2}{\Gamma - X} \pm \sqrt{\frac{Y_T^4}{(\Gamma - X^2)} + 4 Y_L^2} \right], \tag{183}$$

or, transposing,

$$\frac{X}{M^2} = - \left[\Gamma - \frac{Y_T^2}{2(\Gamma - X)} \mp \sqrt{\frac{Y_T^4}{4(\Gamma - X)^2} + 4 Y_L^2} \right]. \tag{184}$$

Taking the reciprocal, multiplying by X, and eliminating M^2,

$$M^2 = n^2 - 1 = - \frac{X}{\Gamma - \frac{Y_T^2}{2(\Gamma - X)} \mp \sqrt{\frac{Y_T^4}{4(\Gamma - X)^2} + Y_L^2}} . \tag{185}$$

Solving for n^2 and substituting from equations (157) and (159),

$$n^2 = 1 - \frac{\omega_p^2 / \omega^2}{\left[1 - j\nu/\omega - \frac{(\omega_b/\omega)^2 \sin^2 \psi}{2(1 - \omega_p^2/\omega^2 - j\nu/\omega)} \right] \pm \left[\frac{\omega_b^4/\omega^4 \sin^4 \psi}{4(1 - \omega_p^2/\omega^2 - j\nu/\omega)^2} + \frac{\omega_b^2}{\omega^2} \cos^2 \psi \right]^{1/2}} . \tag{186}$$

Equation (186) is the well known Appleton equation with ψ being the angle between the geodesic field and the direction of propagation. It is not restricted to any coordinate system. That is, it is a mathematical invariant.

6. FARADAY ROTATION

6.1 Waves Through the Ionosphere

Waves passing through the ionosphere are, generally speaking, broken into two or three distinct waves having distinct indices of refraction, and they travel by distinct ray paths. If the waves are attenuated, they are unlikely to recombine into linearly polarized waves.

However, if attenuation is negligible, the circularly polarized waves may recombine into linearly polarized waves, but they will have their plane of polarization rotated from the original plane. This is referred to as Faraday rotation.

To examine this rotation analytically, consider two such waves emerging from the ionosphere after undergoing different path length shifts in phases, ϕ_1, ϕ_2, respectively. Write the phasor equations,

$$E = E_0 e^{-jk_0 r_0} \left[e^{-jk_0 \int \bar{n}_1 \cdot d\bar{s}_1} + e^{-jk_0 \int \bar{n}_2 \cdot d\bar{s}_2} \right] \tag{187}$$

Since the ionosphere is homogeneous by hypothesis,

$$E = E_0 e^{-jk_0 r_0} (e^{-jk_0 n_1 s_1} + e^{-jk_0 n_2 s_2}) \tag{188}$$

or,

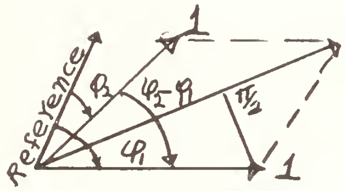
$$E = E_0 e^{-jk(r_0 + \frac{n_1 s_1 + n_2 s_2}{2})} \left[e^{\frac{-jk_0}{2}(n_1 s_1 - n_2 s_2)} + e^{\frac{jk_0}{2}(n_1 s_1 - n_2 s_2)} \right] \tag{189}$$

in which s_1 and s_2 are the respective path lengths.

Upon multiplying and dividing by 2,

$$E = 2 E_0 \cos \left[\frac{k_0}{2} (n_1 s_1 - n_2 s_2) \right] e^{-jk_0[r_0 + 1/2 (n_1 s_1 + n_2 s_2)]}. \tag{190}$$

Thus, the resultant of the shifted vector is $2E_0 \cos [\frac{k_0}{2} (n_1 s_1 - n_2 s_2]$, and $\frac{k_0}{2} (n_1 s_1 - n_2 s_2)$ is the angle of the resultant. This may be verified by referring to figure 9.



$$R = 2 \cos \frac{\varphi_1 \varphi_2}{2}$$

$$\varphi_1 = \frac{k_0}{2} n_1 s_1$$

$$\varphi_2 = \frac{k_0}{2} n_2 s_2$$

Figure 9

From equation (190), it may be seen that the phase shift of the field through the medium is determined by the average path length of the two ray paths.

6.2 Polarization

In order to determine a measure of the complex polarization of a wave propagating within a homogeneous anisotropic ionosphere, it is desirable to find the ratios of the electric field components as determined by equation (179). The matrix form is,

$$\frac{M^2}{X}\begin{bmatrix} \Gamma + \frac{X}{M^2} & -jY_z & \frac{-jY_y}{M^2} \\ jY_z & \Gamma + \frac{X}{M^2} & +\frac{jY_x}{M^2} \\ -jY_y & jY_x & -\frac{(\Gamma - X)}{M^2} \end{bmatrix} \cdot \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0. \qquad (191)$$

From the theory of linear homogeneous equations, the ratio $E_x : E_y : E_z$ may be found from either pair of the three equations, given by equation (191), by omitting the first, second, and third columns of coefficients, respectively. The signs are alternately plus, minus, plus. Thus,

$$E_x : E_y : E_z = \begin{vmatrix} \Gamma + \frac{X}{M^2} & \frac{jY_x}{M^2} \\ jY_x & -\frac{(\Gamma - X)}{M^2} \end{vmatrix} : - \begin{vmatrix} jY_x & \frac{jY_x}{M^2} \\ -jY_y & -\frac{(\Gamma - X)}{M^2} \end{vmatrix} : \begin{vmatrix} jY_z & \Gamma + \frac{X}{M^2} \\ -jY_y & jY_x \end{vmatrix} . \qquad (192)$$

The polarization R is usually defined in terms of the ratio of two components of the electric field normal to the direction of phase propagation. To facilitate the algebra, let,

$$Y_x = 0 , \; Y_z = Y_L , \text{ and } Y_y = Y_T \quad . \qquad (193)$$

This, in effect, rotates the coordinate axes (figure 8) such that $\bar{B}_0$ lies in the Yz - plane. Now write the ratio,

$$R \equiv \frac{E_x}{E_y} = - \frac{\begin{vmatrix} \Gamma + \frac{X}{M^2} & jY_x \\ jY_x & -(\Gamma - X) \end{vmatrix}}{\begin{vmatrix} jY_z & jY_x \\ -jY_y & -(\Gamma - X) \end{vmatrix}} = - \frac{(\Gamma - X)\left(\Gamma + \frac{X}{M^2}\right) - Y_x^2}{j(\Gamma - X)Y_z + Y_x Y_y} , \qquad (194)$$

which, upon substituting from equations (183) and (193), becomes

$$R = \frac{j}{2Y_L}\left[\frac{Y_T^2}{\Gamma - X} \mp \sqrt{\frac{Y_T^4}{(\Gamma - X)^2} + 4Y_L^2}\right] = \frac{-1}{jY_L}\left(\Gamma + \frac{X}{M^2}\right) \quad . \qquad (195)$$

The other ratios may be expressed,

$$\frac{E_z}{E_x} = \frac{-jY_T\ M^2}{\Gamma - X} \quad , \quad \frac{E_z}{E_y} = \frac{-jY_T\ M^2}{\Gamma - X}\ R. \qquad (196)$$

7. CONCLUSION

7.1 Equivalence of Two Points of View

For the purpose of yielding a better insight into the mechanics involved when a wave is propagated into the ionosphere, the principles of electron ballistics were applied to the free electrons. In particular, equation (36) was solved by an iterative procedure, in the procedure for deriving equation (68), the latter equation yielding the dielectric tensor.

The components of the dielectric tensor were used in equation (139) for determining the indices of refraction. This dispersion equation can be formally written,

$$[n^2\, \bar{\bar{I}} - (\bar{\bar{n}} + \bar{\bar{\epsilon}}')] \cdot \bar{E} = 0, \qquad (197)$$

in which an index tensor $\bar{\bar{n}}$ is introduced, with the components of $\bar{\bar{n}}$ being defined as,

$$\bar{\bar{n}} = [n_{ij}] \equiv [n_i\, n_j] \quad , \; i, j = x, y, z. \qquad (198)$$

Equation (36) was also used as a key equation of constraint in deriving the Appleton equation (186). Thus equation (36) may be thought of as a sort of common denominator between the two procedures for finding the complex indices of refraction.

In fact, equation (68) can be derived much more compactly by formalized procedures. For this purpose, consider the corresponding complex form of equation (36),

$$- e\,\bar{E} = (j\omega m + \nu m)\,\bar{V} + e\,\bar{V} \times \bar{B}_0 \qquad (199)$$

Referring to figure 5, and substituting

$$-jY = \frac{e\ B_o}{j\ \omega\ m}\ , \tag{200}$$

equation (199) becomes,

$$-e\ \bar{E} = j\ \omega\ m\ [(1 - j\ \nu/\omega)\ \bar{V} - j\ Y\ \bar{V} \times \bar{a}_z\ ] = j\ \omega\ m\ \times$$

$$[\Gamma\ \bar{V} + j\ Y\ V_x\ \bar{a}_y\ - j\ Y\ V_y\ \bar{a}_x\ ]\ . \tag{201}$$

Hence,

$$\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \frac{-j\ \omega\ m}{e} \begin{bmatrix} \Gamma & -jY & 0 \\ jY & \Gamma & 0 \\ 0 & 0 & \Gamma \end{bmatrix} \cdot \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = \frac{-1}{j\omega\ \epsilon_o\ X} \times$$

$$\begin{bmatrix} \Gamma & -jY & 0 \\ jY & \Gamma & 0 \\ 0 & 0 & \Gamma \end{bmatrix} \cdot \begin{bmatrix} \iota_x \\ \iota_y \\ \iota_z \end{bmatrix}\ . \tag{202}$$

From equation (202), the Lorentz resistivity tensor $\bar{\bar{\rho}}$ may be written, namely,

$$\bar{\bar{\rho}} = \frac{-1}{j\omega\ \epsilon_o X} \begin{bmatrix} \Gamma & -jY & 0 \\ jY & \Gamma & 0 \\ 0 & 0 & \Gamma \end{bmatrix}\ . \tag{203}$$

Now, the complex dielectric tensor is given by,

$$\bar{\bar{\epsilon}}' = \bar{\bar{I}} + j\ \omega\epsilon_o\ (\bar{\bar{\rho}})^{-1}\ . \tag{204}$$

Thus, the inverse resistivity tensor may be found by customary matrix algebra,

$$\bar{\bar{\rho}}^{-1} = \frac{-j\omega\epsilon\ X}{\Gamma(\Gamma^2 - X^2)} \begin{bmatrix} \Gamma^2 & j\Gamma Y & 0 \\ -j\Gamma Y & \Gamma^2 & 0 \\ 0 & 0 & \Gamma^2 - Y^2 \end{bmatrix}, \tag{205}$$

or, $$\bar{\bar{\rho}}^{-1} = -j\omega\ \epsilon_o \begin{bmatrix} \frac{\Gamma X}{X^2 - \Gamma^2} & \frac{jXY}{\Gamma^2 - X^2} & 0 \\ \frac{-jXY}{X^2 - \Gamma^2} & \frac{\Gamma X}{X^2 - \Gamma^2} & 0 \\ 0 & 0 & \frac{X}{\Gamma} \end{bmatrix}\ . \tag{206}$$

Substituting into equation (204) and introducing symbols (63) and (64),

$$\bar{\bar{\epsilon}}\,' = \begin{bmatrix} 1-\dfrac{\Gamma X}{X^2-\Gamma^2} & \dfrac{-jXY}{X^2-\Gamma^2} & 0 \\ \dfrac{jXY}{X^2-\Gamma^2} & 1-\dfrac{\Gamma X}{X^2-\Gamma^2} & 0 \\ 0\;0 & 0 & 1-\dfrac{X}{\Gamma} \end{bmatrix} \equiv \begin{bmatrix} \tilde{X} & -j\tilde{Y} & 0 \\ j\tilde{Y} & \tilde{X} & 0 \\ 0 & 0 & \tilde{Z} \end{bmatrix} . \qquad (207)$$

Substituting $\bar{\bar{\epsilon}}'$ into

$$\text{curl}\ \bar{H} = j\,\omega\epsilon_0\,\bar{\bar{\epsilon}}\,'\bar{E}\ , \qquad (208)$$

yields precisely equation (68), that is, $\dfrac{1}{j\,\omega\,\epsilon_0}$ curl $\bar{H}$ =

$$\bar{a}_x\,(E_x\,\tilde{X} - jE_y\,\tilde{Y}) + \bar{a}_y\,(\,jE_x\,\tilde{Y} + E_y\,\tilde{X}) + \bar{a}_z\,\tilde{Z}\ . \qquad (68)$$

The above derivation tends to place more confidence in the previous physical interpretations. It also serves to tie the iterative procedure to the Appleton equation.

7.2 Inhomogeneous Anisotropic Ionosphere

There is no exact mathematical model for the inhomogeneous anisotropic ionosphere. Many statistical measurements and mathematical interpretations have been and are being made. A vast amount of literature exists, but various precise studies remain to be made.

A detailed study of the inhomogeneous ionosphere was entirely beyond the scope of the time and facilities available for the preparation of this report.

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It is shown that ion collisions introduce a slight reduction in the plasma frequency, along with an exponential decay of transient electron oscillations. The critical frequencies for penetration of a homogeneous ionosphere, for both isotropic and anisotropic ionospheres, are determined. The characteristic waves, for electromagnetic propagation within a homogeneous anisotropic ionosphere, are developed by considering an infinite series of electron velocities, produced by an exciting electric field. The complex indices of refraction are determined, both from a dispersion equation and from a derivation of the Appleton equation, which uses an arbitrary selection of the coordinate axes, thus emphasizing the invariance of the Appleton equation. Vector and tensor algebra is used throughout the analytical developments.

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